Combinatorics and Number Theory

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1. Riemann zeta function

The Riemann zeta function is

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + \cdots)$$
$$= \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \quad \text{for } \Re s > 1.$$

- $\zeta(s)$ has a meromorphic continuation to the whole s-plane;
- It satisfies a functional equation relating $\zeta(s)$ to $\zeta(1-s)$;
- It vanishes at $s = -2, -4, \ldots$, called trivial zeros of ζ.

Riemann Hypothesis: all nontrivial zeros of $\zeta(s)$ lie on the line of symmetry $\Re(s) = \frac{1}{2}$.

2. Zeta functions of curves

X : smooth irred. proj. curve of genus g defined over $\mathbb{F}_q$.
$N_n = \#X(\mathbb{F}_{q^n})$ for each $n \geq 1$
The zeta function of X was introduced by F. K. Schmidt in 1931:

$$\begin{aligned} Z(X;u) = \quad & \exp\left(\sum_{n\geq 1} \frac{N_n}{n} u^n\right) \quad = \quad \frac{P(u)}{(1-u)(1-qu)} \\ = \quad & \prod_{v \text{ closed points of } X} \frac{1}{1-u^{\deg v}}, \end{aligned}$$

where $P(u) = (1-\alpha_1 u)\cdots(1-\alpha_{2g}u) \in \mathbb{Z}[u]$.

Riemann Hypothesis: $|\alpha_i| = q^{1/2}$ for $i = 1, ..., 2g$, proved by Hasse (1933) for $g = 1$ and Weil (1948) in general.

Set $u = q^{-s}$, then RH says all zeros of $Z(X;q^{-s})$ lie on $\Re s = \frac{1}{2}$.

3. The Ihara vertex zeta function of a graph

- X : finite connected undirected graph
- Count backtrackless and tailless cycles.

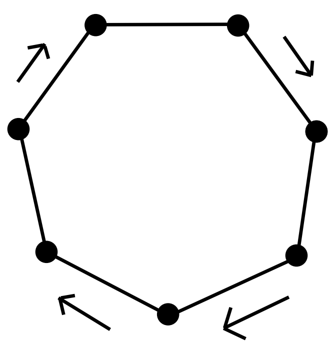

Figure 1: tailless

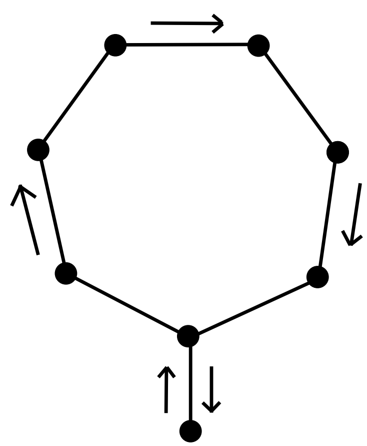

Figure 2: with tail

- *Primitive* cycle: not repeating another cycle more than once.

The Ihara vertex zeta function of X is defined as

$$Z_0(X;u) = \prod_{[C]} \frac{1}{1-u^{l(C)}} \ ,$$

where $[C]$ runs through all equiv. classes of primitive backtrackless and tailless cycles C, and $l(C)$ is the length of C.

Note that

$$u\frac{d}{du}\log Z_0(X;u) = \sum_{\text{all } C} u^{l(C)} = \sum_{n\geq 1} N_n u^n.$$

Here N_n is the number of backtrackless and tailless cycles of length n. Therefore

$$Z_0(X;u) = \prod_{[C]} \frac{1}{1-u^{l(C)}} = \exp\left(\sum_{n\geq 1} \frac{N_n}{n} u^n\right).$$

4. Properties of vertex zeta functions of regular graphs

Theorem [Ihara 1968] *Let X be a finite $(q+1)$-regular graph. Then its zeta function $Z_0(X,u)$ is a rational function of the form*

$$Z_0(X;u) = \frac{(1-u^2)^{\chi(X)}}{\det(I - Au + qu^2 I)} ,$$

where $\chi(X) = \#V - \#E$ is the Euler characteristic of X and A is the vertex adjacency matrix of X.

If X is not regular, replace qI by Q, the degree matrix minus the identity matrix on vertices–Bass, Stark-Terras, Hoffman.

- The trivial eigenvalues of X are $\pm(q+1)$, of multiplicity one.
- X is called a *Ramanujan graph* if the nontrivial eigenvalues λ satisfy the bound
$$|\lambda| \leq 2\sqrt{q},$$
i.e. the roots of $1 - \lambda u + qu^2$ have absolute value $q^{-1/2}$.

  Alon-Boppana: This eigenvalue bound is best possible.

  Explicit constructions of infinite families of Ramanujan graphs for $q = p^a$: Margulis (1988), Lubotzky-Phillips-Sarnak (1988), Mestre-Oesterlé (1986), Pizer (1990), Morgenstern (1994)

  Friedman (2003) : A random regular graph is close to being a Ramanujan graph.
- X is Ramanujan if and only if $Z_0(X, u)$ satisfies RH, i.e. the nontrivial poles of $Z_0(X, u)$ all have absolute value $q^{-1/2}$.

5. The Hashimoto edge zeta function of a graph

Endow two orientations on each edge of a finite graph X. The neighbors of $\overrightarrow{e}$ are the directed edges starting from the ending vertex of $\overrightarrow{e}$ and not equal to the opposite of $\overrightarrow{e}$.

Associate the edge adjacency matrix A_e.

The Hashimoto edge zeta function $Z_1(X,u)$ counts backtrackless and tailless oriented edge cycles, hence the same as $Z_0(X,u)$. Since $N_n = \mathrm{Tr} A_e^n$, we get

$$Z_0(X,u) = Z_1(X,u) = \frac{1}{\det(I - A_e u)}.$$

Combined with Ihara's Theorem, we have

$$(1-u^2)^{\chi(X)} = \frac{\det(I - Au + qu^2 I)}{\det(I - A_e u)}.$$

6. Connections with number theory and group theory

When q is a prime p (or a prime power), the $(p+1)$-regular tree

$$\mathcal{T} \quad = \quad \mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p).$$

$$\begin{array}{rcl}
\text{vertices} & \leftrightarrow & \mathrm{PGL}_2(\mathbb{Z}_p)\text{-cosets} \\
\text{vertex adjacency operator } A & \leftrightarrow & \text{Hecke operator } T_p \text{ on} \\
 & & \mathrm{PGL}_2(\mathbb{Z}_p)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\mathrm{PGL}_2(\mathbb{Z}_p) \\
\text{directed edges} & \leftrightarrow & \mathcal{I}\text{-cosets} \\
\text{edge adjacency operator } A_e & \leftrightarrow & \text{Iwahori-Hecke operator } I_p \text{ on} \\
 & & \mathcal{I}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\mathcal{I}, \text{ where}
\end{array}$$

$$\mathcal{I} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Z}_p) \mid c \in p\mathbb{Z}_p \}$$

is the Iwahori subgroup.

(I) Connection with groups:

Let Γ be a discrete torsion-free cocompact subgroup of $\mathrm{PGL}_2(\mathbb{Q}_p)$. Then $X_\Gamma = \Gamma \backslash \mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$ is finite and $(p+1)$-regular.

Ihara (1968): Γ is a free group on $1+(p-1)|X_\Gamma|/2 = 1-\chi(X_\Gamma)$ generators.

Remark. Originally Ihara defined the zeta function as a product over the hyperbolic primitive conjugacy classes of Γ. Serre noticed the connection with the graph X_Γ.

(II) Connection with zeta function of curves:

When $\Gamma = \Gamma_\ell$ is a suitably chosen subgroup of the quaternion algebra ramified only at ∞ and a prime $\ell \neq p$, we have $\det(I - Au + pu^2 I)/(1-u)(1-pu)$ is the numerator of the zeta function of the modular curve $X_0(\ell)$ mod p by Eichler.

(III) Connection with modular forms:

The graph X_{Γ_ℓ} is Ramanujan
$\Leftrightarrow$ Hecke operator T_p on $S_2(\Gamma_0(\ell))$ satisfies the Ramanujan conjecture
$\Leftrightarrow$ All nontrivial eigenvalues of the Iwahori operator I_p on $S_2(\Gamma_0(\ell p))$ have absolute value $\sqrt{p}$
$\Leftrightarrow$ $Z(X_{\Gamma_\ell}, u)$ satisfies RH.

7. The building associated to $\mathbf{PGL}_3$

Joint work with Ming-Hsuan Kang

- $G = \mathrm{PGL}_3(\mathbb{Q}_p)$, $K = \mathrm{PGL}_3(\mathbb{Z}_p)$
- The Bruhat-Tits building $\mathcal{B} = G/K$ is a 2-dim'l simplicial complex. The chambers are the 2-simplices, their edges are the 1-simplices, and the vertices are the 0-simplices.
- $\mathcal{B}$ is $(p+1)$-regular, namely, each edge is shared by exactly $p+1$ chambers.

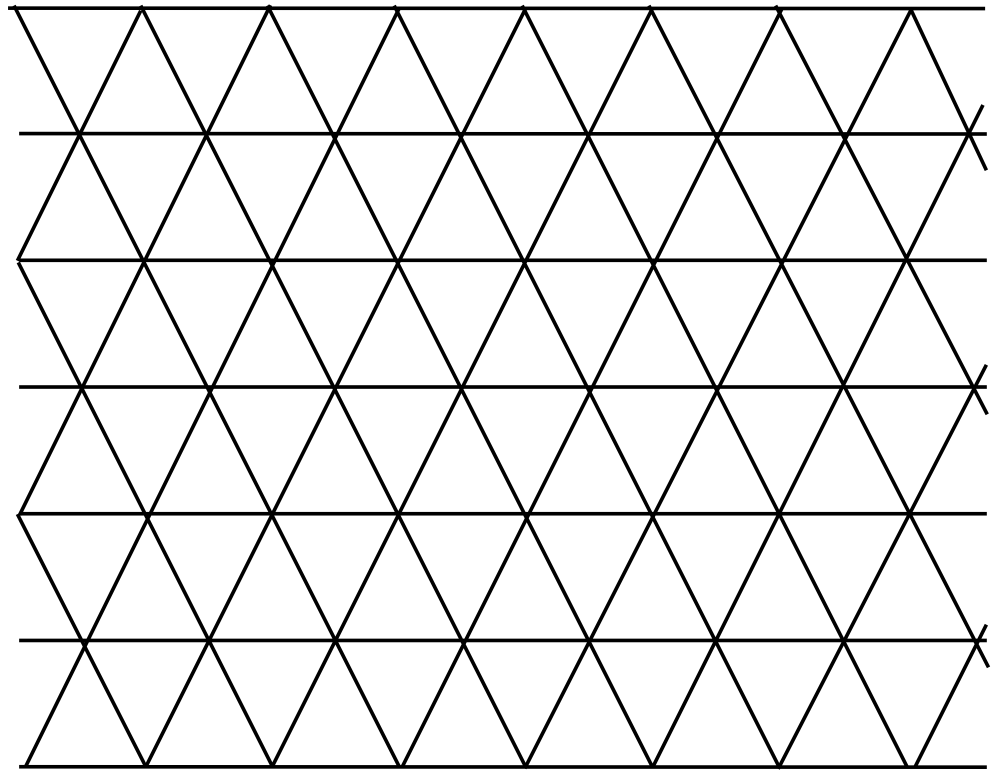

Figure 3: the fundamental apartment

- Topologically $\mathcal{B}$ is simply connected, so it is the universal cover of its finite quotients, called 3-hypergraphs/2-dim'l complexes.

Goal: Find a closed form expression for the vertex zeta function of a finite quotient of $\mathcal{B}$, analogous to the graph zeta function.

Previously considered by Deitmar (2000, 2006) and Deitmar-Hoffman (2006), partial results.

8. Parametrizations of the simplices in $\mathcal{B}$

- $\sigma = \begin{pmatrix} & 1 & \\ & & 1 \\ p & & \end{pmatrix}$. Have a filtration of K:

$$K \supset E := K \ \cap \ \sigma K \sigma^{-1} \supset B := K \ \cap \ \sigma K \sigma^{-1} \cap \ \sigma^{-1} K \sigma.$$

- vertices $\leftrightarrow$ K-cosets

  Each vertex gK has a *type* in $\mathbb{Z}/3\mathbb{Z}$ given by $\tau(gK) := \mathrm{ord}_p(\det g)$ mod 3.

- The type of an edge $gK \longrightarrow g'K$ is $\tau(g'K) - \tau(gK) = 1$ or 2.

- type one edges $\leftrightarrow$ E-cosets

- chambers $\leftrightarrow$ B-cosets such that gB, $g\sigma B$ and $g\sigma^2 B$ represent the same chamber.

9. Hecke operators on $\mathcal{B}$

The K-double cosets define Hecke operators acting on $L^2(G/K)$. They are polynomials in

$$A_1 = K \begin{pmatrix} 1 & & \\ & 1 & \\ & & p \end{pmatrix} K = \bigcup g_i K$$

and

$$A_2 = K \begin{pmatrix} 1 & & \\ & p & \\ & & p \end{pmatrix} K = \bigcup h_j K.$$

The type one edges out of gK are $gK \to gg_iK$ and the type two edges out of gK are $gK \to gh_jK$.

10. Iwahori-Hecke operators on $\mathcal{B}$

The B-double cosets define Iwahori-Hecke operators acting on $L^2(G/B)$. The B-double cosets of G are represented by the Weyl group $W \ltimes < \sigma >$, where W is generated by the three reflections

$$t_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} & & p^{-1} \\ & 1 & \\ p & & \end{pmatrix}, \quad \text{and} \quad t_3 = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}.$$

11. Finite quotients of $\mathcal{B}$

Γ : a discrete torsion-free subgroup of G with compact quotient. Then Γ acts on $\mathcal{B}$ with no fixed points.

$X = X_\Gamma = \Gamma \backslash G/K = \Gamma \backslash \mathcal{B}$

Two assumptions on Γ:

(I) $\mathrm{ord}_p \det \Gamma \subset 3\mathbb{Z}$ so that Γ identifies vertices of the same type.

(II) Γ is *regular*, that is, the centralizer in Γ of any nonidentity element $\gamma \in \Gamma$ is a torus.

Division algebras of degree 9 yield many such Γ's.

12. Homotopy cycles and closed galleries in X

- $\kappa_\gamma(gK)$: the homotopy class of the backtrackless paths from gK to γgK in $\mathcal{B}$ and its image in X, where $\gamma \in \Gamma$.
- Suppose $g^{-1}\gamma g \in K \begin{pmatrix} 1 & & \\ & p^m & \\ & & p^{m+n} \end{pmatrix} K$. Say $\kappa_\gamma(gK)$ has geometric length $n+m$ and algebraic length $n+2m$. When $m=0$ or $n=0$, we say $\kappa_\gamma(gK)$ has *type one* or *two*, accordingly.

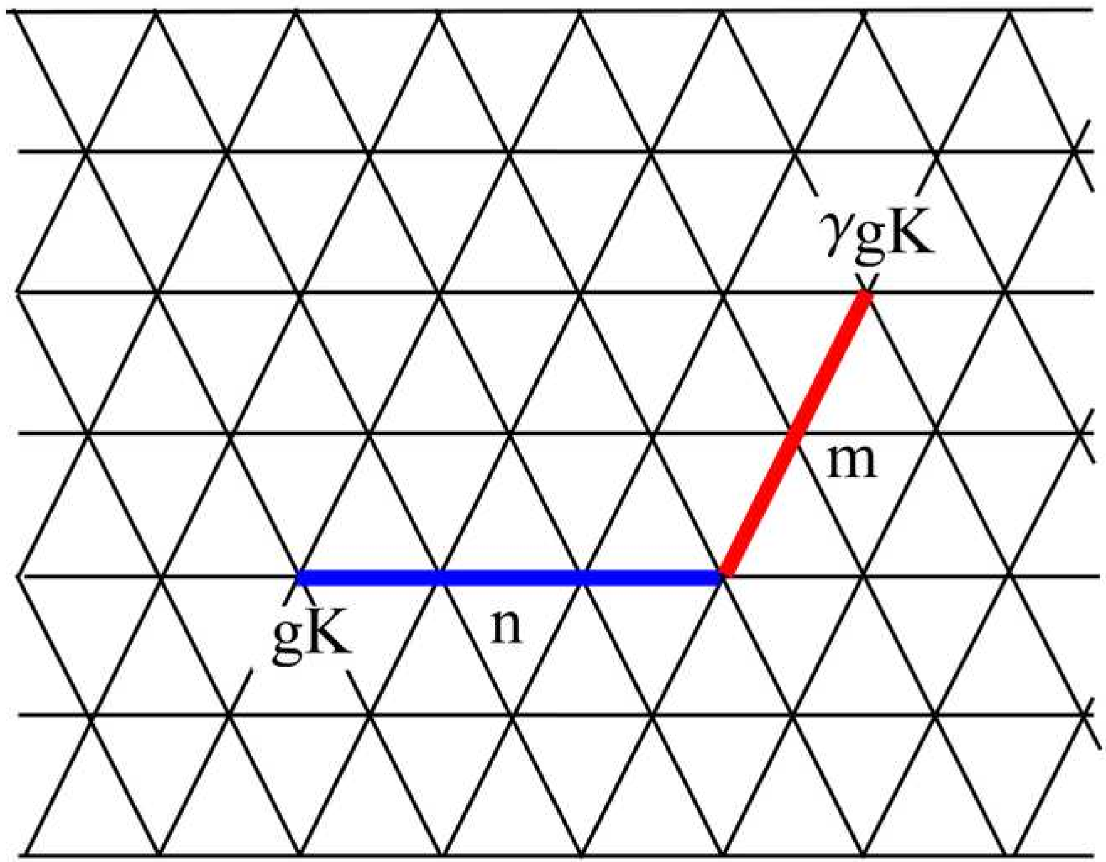

Figure 4: $\kappa_\gamma(gK)$

- A cycle $\kappa_\gamma(gK)$ is called *tailless* if $\kappa_\gamma(hK)$ has the same geometric length as $\kappa_\gamma(gK)$ for all vertices hK lying on the cycle $\kappa_\gamma(gK)$. Define primitive and equivalence as before.

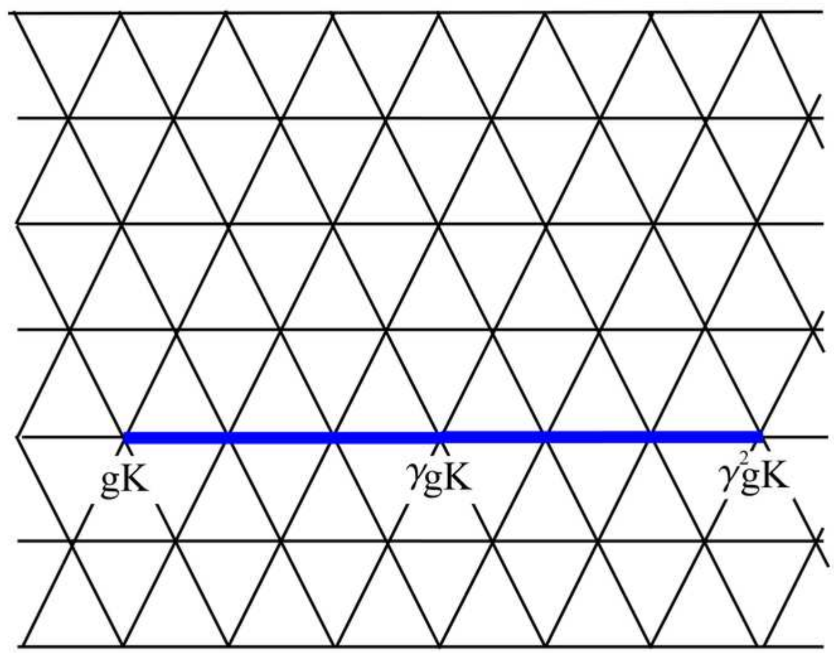

Figure 5: tailless

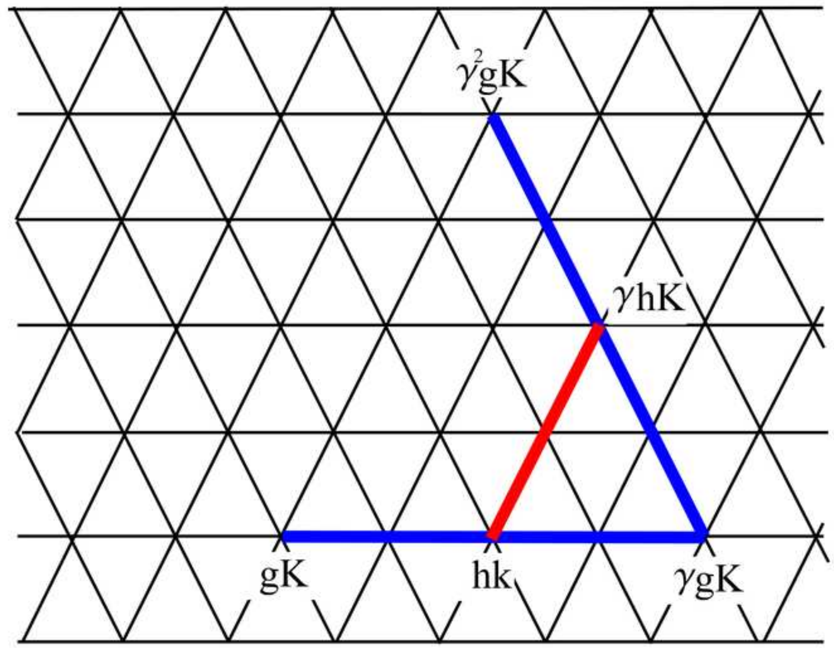

Figure 6: with tail

A sequence of edge-adjacent chambers is called a gallery. The least number of reflections needed to go from the chamber gB to the chamber γgB is the length of the backtrackless gallery $\kappa_\gamma(gB)$.

Interested in tailless type one backtrackless closed galleries in X. Such a gallery has length $n = 3m$ and is uniquely represented by $\kappa_\gamma(gB)$, resulting from repeating m times the reflection sequence $t_2t_1t_3$.

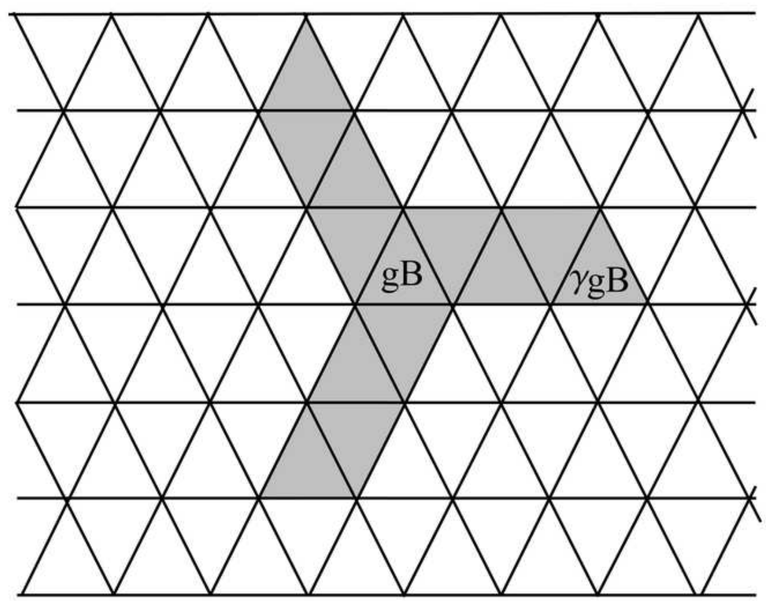

Figure 7: three type one galleries from a chamber

13. Type one chamber zeta function of X

The chamber zeta function of X is defined as

$$Z_2(X,u) = \prod_{[\mathfrak{G}]} \frac{1}{1-u^{l(\mathfrak{G})}} \ ,$$

where $[\mathfrak{G}]$ runs through the equiv. classes of backtrackless primitive tailless type one closed galleries in X.

Let

$$L_B = L_{Bt_2\sigma^2 B}.$$

Theorem. $Z_2(X,u)$ *is a rational function, given by*

$$Z_2(X,u) = \frac{1}{\det(I - L_B u)}.$$

14. Type one edge zeta function of X

The type one edge zeta function is defined as

$$Z_1(X,u) = \prod_{[\mathfrak{C}]} \frac{1}{1-u^{l_A(\mathfrak{C})}} = Z_{0,1}(X,u),$$

where $[\mathfrak{C}]$ runs through the equiv. classes of backtrackless primitive tailless type one cycles in X. This is also the type one vertex zeta function $Z_{0,1}(X,u)$.

Theorem. $Z_1(X,u)$ *is a rational function, given by*

$$Z_1(X_\Gamma,u) = \frac{1}{\det(I - L_E u)}.$$

Here L_E is an operator on $L^2(G/E)$ given by the E-double coset $E(t_2\sigma^2)^2E$. It may be regarded as the "edge adjacency matrix" on the set of type one edges $\Gamma\backslash G/E$ of X_Γ such that the neighbors of a type one edge $v \to v'$ are the p^2 type one edges $v' \to v''$ with v'' not adjacent to v.

15. The vertex zeta function of X

Note that the type one vertex cycles traveled in reverse direction are the type two cycles. Define

$$Z_0(X,u) = Z_{0,1}(X,u)Z_{0,2}(X,u).$$

Main Theorem *$Z_0(X,u)$ is a rational function given by*

$$Z_0(X,u) = \frac{(1-u^3)^{\chi(X)}}{\det(I - A_1 u + pA_2 u^2 - p^3 u^3 I)\det(I + L_B u)},$$

where $\chi(X) = \#V - \#E + \#C$ is the Euler characteristic of X.

Equivalent identity:

$$(1-u^3)^{\chi(X)} = \frac{\det(I - A_1 u + pA_2 u^2 - p^3 u^3 I)\det(I + L_B u)}{\det(I - L_E u)\det(I - L_E u^2)}.$$

- The trivial eigenvalues of $I - A_1u + pA_2u^2 - p^3u^3I$ are 1, p^{-1}, p^{-2} as well as their multiples by the cubic roots of 1.
- X is a 2-dim'l *Ramanujan complex* if the nontrivial eigenvalues of $I - A_1u + pA_2u^2 - p^3u^3I$ all have absolute value p^{-1}.

  Li (2004): Such bounds for eigenvalues of A_1 and A_2 are best possible.

  Explicit constructions of infinite families of Ramanujan complexes: Li (2004), Lubotzky-Samuels-Vishne (2005), Sarveniazi (2007)
- 2-dim'l Ramanujan complexes are characterized by their zeta functions satisfying the RH. In this case, the nontrivial roots of $\det(1 + L_Bu)$ have absolute value $p^{-1/2}$, proved by Kang-Li-Wang.

Connection to zeta function of a surface:

- If X corresponds to a modular surface, the nontrivial part of $\det(I - A_1 u + pA_2 u^2 - p^3 u^3 I)$ is the char. poly. of the Frobenius on the second étale cohomology of the surface, as shown in Laumon-Rapoport-Stuhler (1993)

Information on Γ:

- For PGL_2, all non-identity elements in a discrete torsion-free cocompact subgroup Γ have eigenvalues in the base field. For PGL_3, in addition to such elements, Γ also contains elements which have only one eigenvalue in the base field and the remaining two in a quadratic extension of the base field.

16. Strategy of the proof of the Main Theorem

Type one tailless cycles in X arise as vertex cycles, edge cycles, and also as the boundaries of type one tailless closed galleries. The boundary of a type one tailless closed gallery consists of two or one type one tailless cycles up to equivalence, depending on its length being even or odd. Further, there is a way to characterize the chambers contained in type one tailless closed galleries. This allows us to compare $Z_2(X, -u)$ and $Z_1(X, u^2)$.

The most difficult part lies in analyzing the vertex cycles. To do this we give an algebraic criterion for backtrackless tailless cycles, and compute the number of type one cycles of given length, with and without tails.