MATH 401 - Introduction to Real Analysis

Solutions to Quiz # 3

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3 + 2x$. Prove that f has an inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$. Calculate the value of the derivative $Df^{-1}(y)$ at $y = f(-1) = -3$.

Solution: We have $f'(x) = Df(x) = 3x^2 + 2$ for all $x \in \mathbb{R}$. Thus f is differentiable on $\mathbb{R}$ and strictly increasing since $f'(x) > 0$. Hence f is one-to-one. Moreover,

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty$$

Thus $f(\mathbb{R}) = \mathbb{R}$ and f has an inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$. Their derivatives are linked by the relation:

$$Df^{-1}(y) = \frac{1}{Df(x)}.$$

In particular, when $y = f(-1) = -3$, one gets

$$Df^{-1}(-3) = \frac{1}{Df(-1)} = \frac{1}{3(-1)^2 + 2} = \frac{1}{5}.$$

Problem 2. Which of the following functions are convex:

$f(x) = |x|$ defined on $\mathbb{R}$, $\quad g(x) = 1-\sqrt{1-x^2}$ defined on $[-1, 1]$, $\quad h(x) = x^3-2x$ defined on $\mathbb{R}$.

Solution: (i) $f(x) = |x|$ is convex since given any $\alpha > 0$, $\beta > 0$ with $\alpha+\beta = 1$, one has

$$|\alpha x + \beta y| \leq |\alpha x| + |\beta y| \quad \text{by the triangle inequality}$$

Since $\alpha > 0$ and $\beta > 0$, this gives:

$$|\alpha x + \beta y| \leq \alpha|x| + \beta|y|$$

Hence f is convex.

(ii) $g(x) = 1 - \sqrt{1-x^2}$ is continuous on $[-1, 1]$. Moreover,

$$g'(x) = \frac{2x}{2\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}} \quad \text{on } (-1,\ 1).$$

Furthermore,

$$g"(x) = \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2} = \frac{1}{(1-x^2)\sqrt{1-x^2}}$$

Since $g"(x) > 0$ on the interval $(-1,\ 1)$, we conclude that g is convex.

(iii) $h(x) = x^3 - 2x$ is twice differentiable. Indeed, $h'(x) = 3x^2 - 2$, $h"(x) = 6x$ on $\mathbb{R}$. But $h"$ is **negative** on $(-\infty, 0)$. Hence h in **not convex**.

Problem 3. Consider the function $f(x) = x^2$ on the interval $[0, 1]$.

(i) Compute the upper Riemann sum $\overline{S}(\mathcal{P})$ and the lower Riemann sum $\underline{S}(\mathcal{P})$ corresponding to f and the partition $\mathcal{P} = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$.
(ii) Check that $\underline{S}(\mathcal{P}) \leq \int_0^1 x^2 \, dx \leq \overline{S}(\mathcal{P})$.

Solution: Set $y_0 = 0, y_1 = \frac{1}{4}, y_2 = \frac{1}{2}, y_3 = \frac{3}{4}$ and $y_4 = 1$. We know that $f(x) = x^2$ is continuous on $[0, 1]$, moreover the maxima M_k and minima m_k over the intervals $[y_{k-1}, y_k]$ for $k = 1, 2, 3, 4$ are:

$$m_1 = 0,\ M_1 = \frac{1}{16}, \quad m_2 = \frac{1}{16},\ M_2 = \frac{1}{4}, \quad m_3 = \frac{1}{4},\ M_3 = \frac{9}{16}, \quad m_4 = \frac{9}{16},\ M_4 = 1.$$

The lower Riemann sum is:

$$\underline{S}(\mathcal{P}) = \frac{1}{4}\left(0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right) = \frac{14}{64} = \frac{7}{32}.$$

The upper Riemann sum is:

$$\overline{S}(\mathcal{P}) = \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right) = \frac{30}{64} = \frac{15}{32}.$$

(ii)One has

$$\int_0^1 x^2 \, dx = \frac{1}{3}$$

One gets

$$\underline{S}(\mathcal{P}) = \frac{7}{32} < \frac{1}{3} = \int_0^1 x^2 \, dx < \frac{15}{32} = \overline{S}(\mathcal{P}).$$