PENN STATE UNIVERSITY

MATH 401 - Practice problems for midterm I

FALL 2008

1.

- State the definition of a Cauchy sequence.
- *State the definition of* the infimum *of a non-empty set of real numbers which is bounded below.*
- State the Bolzano-Weierstrass Theorem

2. Determine whether the following series are convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^2}{(n+1)(n+3)} \qquad \sum_{n=1}^{\infty} \frac{1}{7^n + 3n} \qquad \sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n!)} \qquad \sum (-1)^n \frac{1}{n^2\ \sqrt{n}}$$

3. Are the following sequence $\{x_n\}$ increasing or decreasing ? Find an upper bound for the sequence, if it exists

$$(a)\ x_n = \frac{n+1}{2^{n+1}} \qquad (b)\ x_n = \frac{\sqrt{n^2-1}}{n} \qquad (c)\ x_n = \sum_{k=0}^{n-1} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k+1)}$$

4.

a) Prove by mathematical induction Bernouilli's inequality which states that

$$(1+x)^n \geq 1 + nx \quad \text{for } x \geq -1 \text{ and for all } n \in \mathbb{N}.$$

b) Prove by mathematical induction that $3 + 11 + \cdots + (8n-5) = 4n^2 - n$

5.

a) Find the *sup, inf, max, min* of the following sets

$$A = (-\infty, 7), \qquad B = \{1 - \frac{1}{2^n} :\ n \in \mathbb{N}\} \quad \text{and} \quad C = \{x \in \mathbb{Q} : 1 - \sqrt{2} \leq x \leq 1 + \sqrt{3}\}$$

b) Find the following limit

$$\lim_{n \to \infty} \left(\frac{2n}{3n^2} + \frac{2n+1}{3n^2} + \ldots + \frac{3n}{3n^2} \right)$$

6.

a) Using the sandwich theorem, find the following limit:

$$\lim_{n\to\infty} \frac{n+\cos(\sqrt{n})}{5n+2}$$

b) Let $\{a_n\}$ be an arbitrary sequence. Which of the following sequences $\{b_n\}$ has a convergent subsequence? Justify your answer if there is a convergent subsequence. Otherwise, provide a counter-example.

$$(i)\ b_n = \frac{1}{1+|a_n|}, \qquad (ii)\ b_n = 3+\cos a_n \qquad (iii)\ b_n = \frac{a_n}{5+a_n}\ (a_n \neq -5)$$

7. Using the $\epsilon - \delta$ definition of limits, prove that

$$(i)\ \lim_{x\to 3^-} \sqrt{9-x^2} = 0 \qquad (ii)\ \lim_{x\to 0^+} \sqrt{x}\cos(\frac{1}{x}) = 0, \qquad (iii)\ \lim_{x\to 0} \frac{1-x}{1+x^2} = 1$$