MATH 403: Classical Analysis I, Fall 2003
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Chapter 1 - Overview

Section 2.1: The Real Numbers
- Axioms of a field
- Ordered sets - Ordered fields
- Bounded sets

Section 2.2: Infinite Decimals
- Infinite decimal expansions
- Real numbers as infinite decimals

Section 2.3: Limits
- N-epsilon definition of limits
- *Comparison theorem for sequences*
- *The squeeze theorem for sequences*

Section 2.4: Basic Properties of Limits

Section 2.5: Upper and Lower Bounds
- Supremum and infimum
- *Least upper bound principle*
- *Monotone convergence theorem*
- *Limsups and liminfs*

Section 2.6: Subsequences
- *Nested intervals lemma*
- *Bolzano-Weierstrass Theorem*

Section 2.7: Cauchy Sequences
- Complete sets
- *Completeness theorem for* $\mathbb{R}$

Section 2.8: Cardinality
- Injections, surjections, bijections
- Countable and uncountable sets - Cardinals

Chapter 3 - Overview

Section 3.1: Convergent Series

- Definition of a convergent series
- Basic examples: the harmonic series, telescoping sums, geometric series.
- Cauchy criterion for series

Section 3.2: Convergence Test for Series

- *Series with positive terms- Bounded sum test*

  A series $\sum_{n=1}^{\infty} a_n$ with positive terms converges if and only if the sequence $(s_n)_{n=1}^{\infty}$ of partial sums is bounded.

- *Divergence test*

  Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. If the sequence $(a_n)_{n=1}^{\infty}$ does not converge to **zero**, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

- *The comparison test for series*

  Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two series of real numbers with $|a_n| \leq b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

  Example: Show that $\sum_{n=1}^{\infty} 2^{-n} \cos n$ converges.

- *Alternating series test*

  Suppose $(a_n)_{n=1}^{\infty}$ is a monotone decreasing sequence of real numbers that converges to zero, and $a_n \geq 0$, for all n. Then, the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

  Example: Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

- *The root test*

  Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Consider the series $\sum_{n=1}^{\infty} a_n$. Then:

  if $\limsup \sqrt[n]{|a_n|} < 1$ then the series converges absolutely

  if $\limsup \sqrt[n]{|a_n|} > 1$ then the series diverges

  if $\limsup \sqrt[n]{|a_n|} = 1$, this test gives no information

  Example: Show that $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

- *The ratio test*

  Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \varrho.$$

  If $\varrho < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges.

  If $\varrho > 1$ or if $\varrho = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

  If $\varrho = 1$ this test gives is inconclusive.

  Example 1: Show that Euler's series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

  Example 2: Show that the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

- *The integral test*

  Suppose that $f(x)$ is a positive, continuous, and monotone decreasing function on the interval $[1, \infty)$. Let $a_n = f(n)$, for $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ converges.

  Example: Show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

Section 3.4: Absolute and Conditional Convergence

- Absolute Convergence

  If $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. The converse is false.

- *Rearrangement theorem*

  Suppose $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent*. Then, for any real number ℓ, there is a rearrangement of the series that converges to ℓ.

Chapter 4 - Overview

Section 4.1: The Euclidean space $\mathbb{R}^n$

- Schwarz inequality
- Triangle Inequality
- Orthonormal sets

Section 4.2: Convergence and completeness in $\mathbb{R}^n$

- Cauchy sequences in $\mathbb{R}^n$
- *Completeness theorem for* $\mathbb{R}^n$

Section 4.3: Closed and open subsets of $\mathbb{R}^n$

Section 4.4: Compact Sets and The Heine-Borel Theorem

- Compact Sets - *The Heine-Borel Theorem*
- *Cantor's intersection theorem*