MATH 401 - Introduction to Real Analysis
Integration: the Riemann Integral

Let f be a function defined on an interval $I = [a, b]$ and let

$$\mathcal{P} = \{a = y_0 < y_1 < \cdots < y_n = b\}$$

be a partition of $[a, b]$.

- The **mesh** of $\mathcal{P}$ is the maximum length of the intervals $[y_{k-1}, y_k]$.
- The **upper Riemann sum** of f with respect to $\mathcal{P}$ is

$$\overline{S}(\mathcal{P}) = \sum_{i=1}^{n} M_k(y_k - y_{k-1}) \quad \text{where} \quad M_k = \sup\{f(x) : y_{k-1} \leq x \leq y_k\}.$$

- The **lower Riemann sum** of f with respect to $\mathcal{P}$ is

$$\underline{S}(\mathcal{P}) = \sum_{i=1}^{n} m_k(y_k - y_{k-1}) \quad \text{where} \quad m_k = \inf\{f(x) : y_{k-1} \leq x \leq y_k\}.$$

As the mesh of $\mathcal{P}$ tends to zero, we expect both Riemann sums to tend to a fixed number ℓ. If this happens, then f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)dx = \ell = \lim_{mesh \to 0} \underline{S}(P) = \lim_{mesh \to 0} \overline{S}(P).$$

Note: If we partition $[a, b]$ into n intervals of equal length $\frac{b-a}{n}$ then for each value of $n \in \mathbb{N}$, the mesh of the corresponding partition $\mathcal{P}_n$ is

$$mesh(\mathcal{P}_n) = \frac{b-a}{n},$$

which tends to zero as $n \to \infty$.
Theorem

1. If f is continuous on $[a, b]$ then it is Riemann integrable.
2. Non-bounded functions on $[a, b]$ are not Riemann integrable
3. Any linear combination of Riemann integrable functions on $[a, b]$ is also Riemann integrable.