MATH 401 - Spring 2009
Practice problems for the final exam

1. Problems from the textbook:
- Problem set 1 (pages 345- 346): # 4, 6.
- Problem set 2 (page 346-347): # 1, 2.
- Problem set 3 (pages 347-348): # 2, 5, 6
- Problem set 4 (page 348-349): # 3, 4, 5.
- Problem set 5: (page 349): # 2.
- Problem set 6 (page 350): # 2, 3.
- Problem set 7 (page 351): # 2, 6.

2. Problem Prove the following statements by induction:

$$1^3 + 2^3 + \cdots n^3 = (1 + 2 + \cdots n)^2$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

3. Problem. Find the following limit.

$$\lim_{n\to\infty} \frac{\sin(2n+1)}{\sqrt{n+1}} \qquad \lim_{n\to\infty} \frac{3^n - \cos(n)}{3^n + 2^n}$$

4. Problem. Using the $\epsilon - \delta$ definition of limit show that

$$\lim_{x\to 0} \frac{x+1}{x-1} = -1$$

5. Problem . Consider the sequences of real numbers given by

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \qquad \text{and} \qquad b_n = a_n\sqrt{2n+1}$$

a) Show that $\{b_n\}$ is decreasing.
c) Find the limit of $\{a_n\}$.
d) Find the radius of convergence of the series

$$\sum_{n=1}^{n} \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} x^n$$

6. Problem. a) Show that, for all $-1 < x < 1$,

$$\sum_{n=0}^{\infty} (-1)^n (n+1) x^n = \frac{1}{(1+x)^2}$$

b) *Show that for all* $-1 < x < 1$,

$$\mathrm{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

c) Justify the equality

$$\frac{\pi}{8} = \frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \frac{1}{14} + \cdots$$

7. Problem. Prove that, for each value of $y > 0$, the equation

$$\mathrm{Arcsin}(1-x) - x^2 y = 0$$

has a solution x lying in the interval $(0,1)$. (Hint: apply the intermediate value theorem).

8.Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|f(x) - f(y)| \leq (x-y)^2 \sin^2(xy) \quad \text{for all} \quad x, y \in \mathbb{R}.$$

Prove that f is a constant function.

9. Problem. Is the following function differentiable at the origin? Justify your answer.

$$f(x) = \begin{cases} \sqrt{x+1} & \text{if} \quad x \geq 0 \\ 1 + x^2 & \text{if} \quad x < 0 \end{cases}$$

10 Problem. Determine the exact interval of convergence of the power series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} (x-1)^n$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n} (x+1)^n$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 x^{3n}}{(2n+1)(2n+3)(2n+5)}$.

11. Problem. Consider

$$f_n(x) = \frac{nx}{1+3n^2x^2}, \qquad f_n = \frac{x^{2n}}{1+x^{2n}}, \qquad f_n = \frac{3+\sin^2(nx)}{\sqrt{n}}$$

for $x \in \mathbb{R}$, $n \in \mathbb{N}$.
a) Find the pointwise limit of each sequence $\{f_n\}$.
b) Does each of the sequences converge uniformly? Justify your answer.