The Optimal Assignment Problem

Task: Given a weighted complete bipartite graph $G = (X \cup Y, X \times Y)$, where edge xy has weight $w(xy)$, find a matching M from X to Y with maximum weight.

In an application, X could be a set of workers, Y could be a set of jobs, and $w(xy)$ could be the profit made by assigning worker x to job y.

By adding virtual jobs or workers with 0 profitability, we may assume that X and Y have the same size, n, and can be written as $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Mathematically, the problem can be stated: given an $n \times n$ matrix W, find a permutation π of $\{1, 2, 3, \ldots, n\}$ for which

$$\sum_{i=1}^{n} w(x_i y_{\pi(i)})$$

is a maximum.

Such a matching from X to Y is called an *optimal assignment*.

Definition: A *feasible vertex labeling* in G is a real-valued function l on $X \cup Y$ such that for all $x \in X$ and $y \in Y$,

$$l(x) + l(y) \geq w(xy).$$

Such labelings may be conveniently written beside the matrix of weights.

It is always possible to find a feasible vertex labeling. One way to do this is to set all $l(y) = 0$ for $y \in Y$ and for each $x \in X$, take the maximum value in the corresponding row, *i.e.*

$$\begin{array}{ll} l(x) = \max_{y \in Y} w(xy) & \text{for } x \in X \\ l(y) = 0 & \text{for } y \in Y \end{array}$$

If l is a feasible labeling, we denote by G_l the subgraph of G which contains those edges where $l(x) + l(y) = w(xy)$, together with the endpoints of these edges. This graph G_l is called the *equality subgraph* for l.

Theorem. If l is a feasible vertex labeling for G, and M is a complete matching of X to Y with $M \subseteq G_l$, then M is an optimal assignment of X to Y.

Proof: We must show that no other complete matching can have weight greater than M. Let any complete matching M' of X to Y be given. Then

$$\begin{aligned} w(M') &= \sum_{xy \in M'} w(xy) \\ &\leq \sum_{xy \in M'} (l(x) + l(y)) \quad \text{(feasibility of } l\text{)} \\ &= \sum_{xy \in M} (l(x) + l(y)) \quad \text{(all the } l(x) \text{ and } l(y) \text{ are summed in either matching)} \\ &= \sum_{xy \in M} w(xy) \quad \text{since } M \subseteq G_l \\ &= w(M). \end{aligned}$$

Thus the problem of finding an optimal assignment is reduced to the problem of finding a feasible vertex labeling whose equality subgraph contains a complete matching of X to Y.

The Kuhn-Munkres Algorithm (also known as the Hungarian method).

Start with an arbitrary feasible vertex labeling l, determine G_l, and choose an arbitrary matching M in G_l.

1. If M is complete for G, then M is optimal. Stop. Otherwise, there is some unmatched $x \in X$. Set $S = \{x\}$ and $T = \emptyset$.

2. If $J_{G_l}(S) \neq T$, go to step 3. Otherwise, $J_{G_l}(S) = T$. Find

$$\alpha_l = \min_{x \in S, y \in T^c} \{l(x) + l(y) - w(xy)\}$$

   where T^c denotes the complement of T in Y, and construct a new labeling l' by

$$l'(v) = \begin{cases} l(v) - \alpha_l & \text{for } v \in S \\ l(v) + \alpha_l & \text{for } v \in T \\ l(v) & \text{otherwise} \end{cases}$$

   Note that $\alpha_l > 0$ and $J_{G_{l'}}(S) \neq T$. Replace l by l' and G_l by $G_{l'}$.

3. Choose a vertex y in $J_{G_l}(S)$, not in T. If y is matched in M, say with $z \in X$, replace S by $S \cup \{z\}$ and T by $T \cup \{y\}$, and go to step 2. Otherwise, there will be an M-alternating path from x to y, and we may use this path to find a larger matching M' in G_l. Replace M by M' and go to step 1.