The Optimal Assignment Problem

**Task:** Given a weighted complete bipartite graph \( G = (X \cup Y, X \times Y) \), where edge \( xy \) has weight \( w(xy) \), find a matching \( M \) from \( X \) to \( Y \) with maximum weight.

In an application, \( X \) could be a set of workers, \( Y \) could be a set of jobs, and \( w(xy) \) could be the profit made by assigning worker \( x \) to job \( y \).

By adding virtual jobs or workers with 0 profitability, we may assume that \( X \) and \( Y \) have the same size, \( n \), and can be written as \( X = \{x_1, x_2, \ldots, x_n\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \).

Mathematically, the problem can be stated: given an \( n \times n \) matrix \( W \), find a permutation \( \pi \) of \( \{1, 2, 3, \ldots, n\} \) for which

\[
\sum_{i=1}^{n} w(x_i y_{\pi(i)})
\]

is a maximum.

Such a matching from \( X \) to \( Y \) is called an optimal assignment.

**Definition:** A feasible vertex labeling in \( G \) is a real-valued function \( l \) on \( X \cup Y \) such that for all \( x \in X \) and \( y \in Y \),

\[
l(x) + l(y) \geq w(xy).
\]

Such labelings may be conveniently written beside the matrix of weights.

It is always possible to find a feasible vertex labeling. One way to do this is to set all \( l(y) = 0 \) for \( y \in Y \) and for each \( x \in X \), take the maximum value in the corresponding row, i.e.

\[
l(x) = \max_{y \in Y} w(xy) \quad \text{for } x \in X
\]

\[
l(y) = 0 \quad \text{for } y \in Y
\]

If \( l \) is a feasible labeling, we denote by \( G_l \) the subgraph of \( G \) which contains those edges where \( l(x) + l(y) = w(xy) \), together with the endpoints of these edges. This graph \( G_l \) is called the equality subgraph for \( l \).

**Theorem.** If \( l \) is a feasible vertex labeling for \( G \), and \( M \) is a complete matching of \( X \) to \( Y \) with \( M \subseteq G_l \), then \( M \) is an optimal assignment of \( X \) to \( Y \).

Proof: We must show that no other complete matching can have weight greater than \( M \). Let any complete matching \( M' \) of \( X \) to \( Y \) be given. Then

\[
w(M') = \sum_{xy \in M'} w(xy)
\]

\[
\leq \sum_{xy \in M'} (l(x) + l(y)) \quad \text{(feasibility of } l)\)
\]

\[
= \sum_{xy \in M} (l(x) + l(y)) \quad \text{(all the } l(x) \text{ and } l(y) \text{ are summed in either matching)}
\]

\[
= \sum_{xy \in M} w(xy) \quad \text{since } M \subseteq G_l
\]

\[
= w(M).
\]
Thus the problem of finding an optimal assignment is reduced to the problem of finding a feasible vertex labeling whose equality subgraph contains a complete matching of $X$ to $Y$.

**The Kuhn-Munkres Algorithm** (also known as the Hungarian method).

Start with an arbitrary feasible vertex labeling $l$, determine $G_l$, and choose an arbitrary matching $M$ in $G_l$.

1. If $M$ is complete for $G$, then $M$ is optimal. Stop. Otherwise, there is some unmatched $x \in X$. Set $S = \{x\}$ and $T = \emptyset$.

2. If $J_{G_l}(S) \neq T$, go to step 3. Otherwise, $J_{G_l}(S) = T$. Find

$$\alpha_l = \min_{x \in S, y \in T^c} \{l(x) + l(y) - w(xy)\}$$

where $T^c$ denotes the complement of $T$ in $Y$, and construct a new labeling $l'$ by

$$l'(v) = \begin{cases} 
    l(v) - \alpha_l & \text{for } v \in S \\
    l(v) + \alpha_l & \text{for } v \in T \\
    l(v) & \text{otherwise}
\end{cases}$$

Note that $\alpha_l > 0$ and $J_{G_{l'}}(S) \neq T$. Replace $l$ by $l'$ and $G_l$ by $G_{l'}$.

3. Choose a vertex $y$ in $J_{G_l}(S)$, not in $T$. If $y$ is matched in $M$, say with $z \in X$, replace $S$ by $S \cup \{z\}$ and $T$ by $T \cup \{y\}$, and go to step 2. Otherwise, there will be an $M$-alternating path from $x$ to $y$, and we may use this path to find a larger matching $M'$ in $G_l$. Replace $M$ by $M'$ and go to step 1.