CHARACTERIZING THE NUMBER OF $m$–ARY PARTITIONS MODULO $m$

GEORGE E. ANDREWS, AVIEZRI S. FRAENKEL, AND JAMES A. SELLERS

Abstract. Motivated by a recent conjecture of the second author related to the ternary partition function, we provide an elegant characterization of the values $b_m(nm)$ modulo $m$ where $b_m(n)$ is the number of $m$-ary partitions of the integer $n$ and $m \geq 2$ is a fixed integer.

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1. Introduction

Congruences for partition functions have been studied extensively for the last century or so, beginning with the discoveries of Ramanujan [7]. In this note, we will focus our attention on congruence properties for the partition functions which enumerate restricted integer partitions known as $m$-ary partitions. These are partitions of an integer $n$ wherein each part is a power of a fixed integer $m \geq 2$. Throughout this note, we will let $b_m(n)$ denote the number of $m$-ary partitions of $n$.

As an example, note that there are five 3-ary partitions of $n = 9$:

$$9, \quad 3 + 3 + 3, \quad 3 + 3 + 1 + 1 + 1,$$
$$3 + 1 + 1 + 1 + 1 + 1 + 1, \quad 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

Thus, $b_3(9) = 5$.

In the late 1960s, Churchhouse [3, 4] initiated the study of congruence properties of binary partitions ($m$-ary partitions with $m = 2$). By his own admission, he did so serendipitously. To quote Churchhouse [4], “It is however salutary to realise that the most interesting results were discovered because I made a mistake in a hand calculation!”

Within months, other mathematicians proved Churchhouse’s conjectures and proved natural extensions of his results. These included Rødseth [8] who extended Churchhouse’s results to include the functions $b_p(n)$ where $p$ is any prime as well as Andrews [2] and Gupta [5, 6] who proved that corresponding results also held for $b_m(n)$ where $m$ could be any integer greater than 1. As part of an infinite family of results, these authors proved that, for any $m \geq 2$ and any nonnegative integer $n$, $b_m(m(mn - 1)) \equiv 0 \pmod m$.

We now fast forward forty years. In 2012, the second author conjectured the following absolutely remarkable result related to the ternary partition function $b_3(n)$:

- For all $n \geq 0$, $b_3(3n)$ is divisible by 3 if and only if at least one 2 appears as a coefficient in the base 3 representation of $n$.

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Moreover, \[ b_3(3n) \equiv (-1)^j \pmod{3} \] whenever no 2 appears in the base 3 representation of \( n \) and \( j \) is the number of 1s in the base 3 representation of \( n \).

This conjecture is remarkable for at least two reasons. First, it provides a complete characterization of \( b_3(3n) \) modulo 3. Such characterizations in the world of integer partitions are rare. Secondly, the result depends on the base 3 representation of \( n \) and nothing else.

Just to “see” what the second author saw, let’s quickly look at some data related to this conjecture.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Base 3 Representation of ( n )</th>
<th>( b_3(3n) )</th>
<th>( b_3(3n) \pmod{3} )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>( 1 \times 3^0 )</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>( 2 \times 3^0 )</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( 0 \times 3^0 + 1 \times 3^1 )</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>( 1 \times 3^0 + 1 \times 3^1 )</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>( 2 \times 3^0 + 1 \times 3^1 )</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>( 0 \times 3^0 + 2 \times 3^1 )</td>
<td>12</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>( 1 \times 3^0 + 2 \times 3^1 )</td>
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<td>0</td>
</tr>
<tr>
<td>8</td>
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<td>18</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>( 0 \times 3^0 + 0 \times 3^1 + 1 \times 3^2 )</td>
<td>23</td>
<td>2</td>
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<tr>
<td>10</td>
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<tr>
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<td>0</td>
</tr>
<tr>
<td>15</td>
<td>( 0 \times 3^0 + 2 \times 3^1 + 1 \times 3^2 )</td>
<td>63</td>
<td>0</td>
</tr>
</tbody>
</table>

In recent days, the authors succeeded in proving this conjecture. Thankfully, the proof was both elementary and elegant. After just a bit of additional consideration, we were able to alter the proof to provide a completely unexpected generalization. We describe this generalized result, and provide its proof, in the next section.

2. The Full Result

Our main theorem, which includes the above conjecture in a very natural way, provides a complete characterization of \( b_m(mn) \) modulo \( m \):

**Theorem 2.1.** Let \( m \geq 2 \) be a fixed integer and let

\[ n = a_0 + a_1 m + \cdots + a_j m^j \]

be the base \( m \) representation of \( n \) (so that \( 0 \leq a_i \leq m - 1 \) for each \( i \)). Then

\[ b_m(mn) \equiv \prod_{i=0}^{j} (a_i + 1) \pmod{m}. \]

Notice that the conjecture mentioned above is exactly the \( m = 3 \) case of Theorem 2.1.

In order to prove Theorem 2.1, we need a few elementary tools. We describe these tools here.
First, it is important to note that the generating function for \( b_m(n) \) is given by

\[
B_m(q) := \prod_{j=0}^{\infty} \frac{1}{1 - q^{mj}}.
\]

Note that \( B_m(q) \) satisfies the functional equation

\[
(1 - q)B_m(q) = B_m(q^m).
\]

From here it is straightforward to prove that

\[
b_m(mn) = b_m(mn+i)
\]

for all \( 1 \leq i \leq m - 1 \). Thus, we see that Theorem 2.1 actually provides a characterization of \( b_m(N) \mod m \) for all \( N \), not just for those \( N \) which are multiples of \( m \).

With this information in hand, we now prove a small number of lemmas which we will use in our proof of Theorem 2.1.

**Lemma 2.2.** For \( |x| < 1 \),

\[
\frac{1 - x^m}{(1 - x)^2} \equiv \sum_{k=1}^{m} kx^{k-1} \pmod{m}.
\]

**Proof.** This elementary congruence can be proven rather quickly using well–known mathematical tools. We begin with the geometric series identity

\[
\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k.
\]

Differentiating both sides yields

\[
\frac{1}{(1 - x)^2} = \sum_{k=1}^{\infty} kx^{k-1}.
\]

We then multiply both sides by \( 1 - x^m \) and simplify as follows:

\[
\frac{1 - x^m}{(1 - x)^2} = \sum_{k=1}^{\infty} kx^{k-1} - x^m \sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=1}^{\infty} kx^{k-1} - \sum_{k=m+1}^{\infty} (k - m)x^{k-1} = \sum_{k=1}^{m} kx^{k-1} + \sum_{k=m+1}^{\infty} m x^{k-1} \equiv \sum_{k=1}^{m} kx^{k-1} \pmod{m}
\]

\[\blacksquare\]

**Lemma 2.3.** Let \( \zeta \) be the \( m \)th root of unity given by \( \zeta = e^{2\pi i / m} \). Then

\[
\sum_{k=0}^{m-1} \frac{1}{1 - \zeta^k q} = \frac{1}{1 - q^m}.
\]
Proof. Using geometric series and elementary series manipulations, we have

\[
\sum_{k=0}^{m-1} 1 - \zeta^k q^r = \sum_{k=0}^{m-1} \sum_{r=0}^{\infty} \zeta^{kr} q^r = \sum_{k=0}^{m-1} 1 - q^r (\sum_{r \mid m} \zeta^{kr} q^r) = \sum_{k=0}^{m-1} 1 - q^r \text{ using facts about roots of unity} = m \left( \frac{1}{1 - q^m} \right).
\]

**Lemma 2.4.** Let \( T_m(q) := \sum_{n \geq 0} b_m(mn)q^n \). Then

\[ T_m(q) = \frac{1}{1 - qB_m(q)}. \]

**Proof.** As in Lemma 2.3, let \( \zeta = e^{2\pi i/m} \). Note that

\[
T_m(q^m) = \sum_{n \geq 0} b_m(mn)q^{mn} = \frac{1}{m} \left( B_m(q) + B_m(\zeta q) + \cdots + B_m(\zeta^{m-1} q) \right) = \left( \prod_{j=1}^{\infty} \frac{1}{1 - q^{mj}} \right) \times \frac{1}{m} \sum_{k=0}^{m-1} \frac{1}{1 - \zeta^k q} = \frac{1}{1 - q^m} \prod_{j=1}^{\infty} \frac{1}{1 - q^{mj}}
\]

thanks to Lemma 2.3. Lemma 2.4 then follows by replacing \( q^m \) by \( q \).

We now combine these elementary facts from the lemmas above to prove one last lemma. This lemma will, in essence, allow us to “move” from considering \( T_m(q) \) modulo \( m \) to a new function modulo \( m \) which makes the result of Theorem 2.1 transparent.

**Lemma 2.5.** Let \( U_m(q) = \prod_{j=0}^{\infty} \left( 1 + 2q^{mj} + 3q^{2mj} + \cdots + mq^{(m-1)mj} \right) \). Then

\[ T_m(q) \equiv U_m(q) \pmod{m}. \]

**Proof.** Lemma 2.5 will follow if we can prove that \( \frac{1}{T_m(q)} : U_m(q) \equiv 1 \pmod{m} \), and this will be our means of attack. Thankfully, this follows from a novel generating function manipulation which we demonstrate here. Using (1) and Lemma 2.4, we
know that
\[
\frac{1}{T_m(q)} \cdot U_m(q)
\]
\[
= (1 - q)^2 \prod_{j=1}^{\infty} (1 - q^{m^j}) \prod_{j=0}^{\infty} \left(1 + 2q^{m^j} + 3q^{2m^j} + \cdots + mq^{(m-1)m^j}\right)
\]
\[
\equiv (1 - q)^2 \prod_{j=1}^{\infty} (1 - q^{m^j}) \prod_{j=0}^{\infty} \frac{1 - q^{m^j+1}}{(1 - q^{m^j})^2} \pmod{m} \quad \text{thanks to Lemma 2.2}
\]
\[
= \prod_{j=0}^{\infty} \frac{1 - q^{m^j+1}}{1 - q^{m^j}}
\]
\[
= 1.
\]

We can now utilize all of the above results to prove Theorem 2.1.

Proof. First, we remember that
\[
\sum_{n \geq 0} b_m(mn)q^n = T_m(q) \equiv U_m(q) \pmod{m}.
\]
So we simply need to consider $U_m(q)$ modulo $m$ to obtain our proof. Note that
\[
U_m(q) = \prod_{j=0}^{\infty} \left(1 + 2q^{m^j} + 3q^{2m^j} + \cdots + mq^{(m-1)m^j}\right).
\]

If we expand this product as a power series in $q$, then each term of the form $q^n$ can occur at most once (because the terms $q^{i \cdot m^j}$ are serving as the building blocks for the unique base $m$ representation of $m$). Thus, if
\[
n = a_0 + a_1m + \cdots + a_jm^j,
\]
then the coefficient of $q^n$ in this expansion is
\[
\prod_{i=0}^{j} (a_i + 1) \pmod{m}.
\]

References
7. S. Ramanujan, Some properties of $p(n)$, the number of partitions of $n$, Proc. Cambridge Philos. Soc. 19 (1919), 207-210