A Witt Group Structure on Orbits of Unimodular Rows

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The Elemantary Symplectic Witt Groups

The ‘addition’ operation $\perp$: $\alpha \perp \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

The basic alternating matrix $\psi_r \in E_{2r}(\mathbb{Z})$:

$$\psi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \psi_r = \psi_{r-1} \perp \psi_1$$

Stable equivalence of alternating matrices w.r.t. $E(A)$ — the stable elementary subgroup of $GL(A)$:

Let $\phi \in M_{2r}(A)$, $\eta \in M_{2s}(A)$, then $\phi \simeq \eta$ iff

$$\phi \perp \psi_{s+t} = \varepsilon^t(\eta \perp \psi_{r+t})\varepsilon,$$

for some $t \geq 0$, $\varepsilon \in E_{2(r+s+t)}(A)$.

$\simeq$ is an equivalence relation of the set of all matrices, and also on the set of all alternating matrices of Pfaffian one.

One can show that $\perp$ defines an addition on the equivalence classes of alternating matrices of Pfaffian one; and this is actually an abelian group $\boldsymbol{W_E(A)}$ — known as the *Elementary Symplectic Witt group.*

$[\psi_r]$ is the identity element, and $[\phi]^{-1} = [\phi^{-1}]$.

The Vaserstein Symbol $V : Um_3(A) \longrightarrow W_E(A)$:

$$[(\mathbf{a},\mathbf{b},\mathbf{c})] \longrightarrow \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c' & b' \\ -b & c' & 0 & a' \\ -c & -b' & -a' & 0 \end{pmatrix}$$

$$= V(a, b, c; a', b', c')$$

$$aa' + bb' + cc' = 1$$

Theorem: (Vaserstein) V *is an isomorphism if Krull dimension* A *is two.*

The Vaserstein rule: Addition of Vectors in Dimension Two

Let $xx' + yy' = 1$ modulo (a), then

$$[(a,b,c)] * [(a,x,y)] = \left[\left(a, (b,c)\begin{pmatrix} x & y \\ -y' & x' \end{pmatrix}\right)\right]$$

Mennicke-Newmann Lemma for Unimodular Vectors

Let $v, w \in Um_n(A)$, with $d = \dim(A) \leq 2n - 3$. There there are $\varepsilon_1, \varepsilon_2 \in E_n(A)$, such that

$$v\varepsilon_1 = (x, a_2, \ldots, a_n)$$
$$w\varepsilon_2 = (y, a_2, \ldots, a_n)$$

Moreover, in fact, one can also arrange that $x + y = 1$.

Counter-example in Dimension Three

Let

$$A = \mathbb{R}[x, y, z, t]/(x^2 + y^2 + z^2 + t^2 - 1)$$

be the 3-dimensional co-ordinate ring of the real 3 sphere S^3. In 1992, W. van der Kallen and I observed that the Vaserstein symbol

$$V\ : Um_3(A)/E_3(A) \longrightarrow W_E(A)$$

is **not** injective. We did this by finding two vectors v, w which were not in the same elementary orbit, but were equal in $W_E(A)$.

Let
$v = (-t^2 + x^2 + y^2 - z^2, -2tx + 2yz, 2ty + 2xz)$
$\in Um_3(A)$. (In fact v is completable: Consider the three dimensional real vector space W of pure quaternions. Let

$$q = x + yi + zj + tk.$$

Then q is a unit quaternion, and acts on W by conjugation: $p \longrightarrow qpq^{-1}$. It can be checked that v is the first row of the matrix corresponding to this linear transformation. This row can be viewed as a map $h : S^3 \longrightarrow S^2$, and is known as the Hopf map - as this map generates $\pi_3(S^2)$: Verify that $(1, 0, 0)$ and $(-1, 0, 0)$ are regular values whose inverse images are two circles which are simply linked in S^3.

The second vector w is got from v by substituting $-z$ for z. Since we are reversing the orientation on S^3, we replace the Hopf map by its negative in $\pi_3(S^2)$; which is different. Hence $[v] \neq [w]$. However, using Vaserstein's Rule it was possible to show that $V([v]) = V([w])$.

Example in Dimension Three

In the same article, W. van der Kallen and I showed that if A is a three dimensional non-singular affine algebra over a perfect C_1 field k then the Vaserstein symbol is an isomorphism.

Van der Kallen's Group Structure

Using L.N. Vaserstein's group structure in dimension two as an inductive step W. van der Kallen first defined a group structure on $Um_{d+1}(A)$, when A had Krull dimension d as follows: Choose p_0 such that $a_0p_0 = 1$ modulo $(a_1, a_2, \ldots, a_d)$. Define

$$[(a_0, a_1, \ldots, a_d)] * [(b_0, a_1, \ldots, a_d)] = \\ [(a_0(b_0 + p_0) - 1, (b_0 + p_0)a_1, a_2 \ldots, a_d)]$$

Note: If $\dim(A) = 2$, $a_0p_0 + a_1p_1 + a_2p_2 = 1$, then

$$(b_0, a_1)\begin{pmatrix} a_0 & a_1 \\ -p_1 & p_0 \end{pmatrix} = (a_0b_0 - a_1p_1, b_0a_1 + p_0a_1)$$
$$= (a_0(b_0 + p_0) - 1 + a_2p_2, (b_0 + p_0)a_1)$$

He constructs the group structure by induction. If modulo (a_d),

$$[(a_0, a_1, \ldots, a_{d-1})] * [(b_0, a_1, \ldots, a_{d-1})] =$$
$$[(c_0, c_1, \ldots, c_{d-1})],$$

then

$$[(a_0, a_1, \dots, a_d)] * [(b_0, a_1, \dots, a_d)] = [(c_0, c_1, \dots, c_{d-1}, a_d)]$$

Weak Mennicke Symbols and Group Structure on Orbits

If $R = C(X)$ is the ring of continuous real valued functions on a topological space X then every unimodular vector $v \in Um_n(C(X)), n \geq 2$, determines a map

$$\arg(v) : X \longrightarrow \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$$

(The first is by evaluation, and the second is the standard homotopy equivalence.) We thus get an element $[\arg(v)]$ of $[X, S^{n-1}]$. (As $n \geq 2$, we may ignore base points.) Clearly, vectors in the same elementary orbit define homotopic maps. Thus, we have a natural map

$$Um_n(C(X))/E_n(C(X)) \longrightarrow [X, S^{n-1}] = \pi^{n-1}(X).$$

Note that J.F. Adams has shown that S^{n-1} is not a H-space, unless $n = 1, 2, 4$, or 8. It is classically known that this is equivalent to saying that there is no suitable way to multiply the two projection maps $S^{n-1} \times S^{n-1}$ in $[S^{n-1} \times S^{n-1}, S^{n-1}]$. However, under suitable restrictions on the 'dimension' of X we may expect to define a product.

Henceforth, let X be a finite CW-complex of dimension $d \geq 2$. L.N. Vaserstein has shown that the ring $C(X)$ has stable dimension d. Now let $n \geq 3$, so that S^{n-1} will be atleast 1-connected. By the Suspension Theorem, the suspension may

$$S : [X; S^{n-1}] \longrightarrow [SX; S^n]$$

is surjective if $d \leq 2(n-2)+1$, and bijective if $d \leq 2(n-2)$. Moreover, we know that $[SX, S^n]$ is an abelian group. Hence, the orbit space has a structure of an abelian group. It can also be shown that above map is a universal weak Mennicke symbol (explained below).

Inspired by the groups structures on orbits of unimodular vectors in the case of rings of continuous functions $C(X)$ on a CW-complex X, W. van der Kallen was able to obtain similar results algebraically, in the same range. He also showed that the above map is an isomorphism of groups in the topological situation when $R = C(X)$ is the ring of continuous real valued functions on a finite CW-complex X of dimension $d \geq 2$.

The main reason this was possible is a natural extension of the Vaserstein Rule described earlier.

Van der Kallen proves directly that there is a group structure on the orbit space. This was done by a patient study of the effect of minor variation in the choices, in about 20 steps. The formulae are similar to the above case of $d+1$-sized vectors.

He then gives a universal weak Mennicke symbol interpretation of the group, which we briefly mention next.

Weak Mennicke Symbols

A weak Mennicke symbol (of order n) over R is a map

$$wms : Um_n(R)/E_n(R) \longrightarrow G,$$

a group G, such that, whenever $(q, v_2, \ldots, v_n)$, $(1+q, v_2, \ldots, v_n)$ are unimodular, and $r(1+q) = q$ modulo $(v_2, \ldots, v_n)$, one has

$$wms(q, v_2, \ldots, v_n) = wms(r, v_2, \ldots, v_n) wms(1+q, v_2, \ldots, v_n).$$

In 1989, W. van der Kallen showed that, if $n \geq 3$, and

$sdim(R) \leq 2n-4$, then the universal weak Mennicke symbol

$$wms : Um_n(R)/E_n(R) \longrightarrow WMS_n(R)$$

is bijective. He also showed that the surjectivity implied that the latter had an abelian group structure.

Remark. *To prove the bijectivity, it was necessary for him to actually prove directly that there is a group structure on the orbit space.*

Remark. *Recently, following the important work of S.M.Bhatwadekar-Raja Sridharan, who constructed a homomorphism from an orbit set of unimodular rows* *to an Euler class group, W. van der Kallen was able to show that the multiplication in the orbit spaces can be succinctly described as follows:*

$$[(a_0(1-a_0), a_1, \ldots, a_n)] = [(a_0, a_1, \ldots, a_n)] * [(1-a_0, a_1, \ldots, a_n)]$$

(Since one can arrange that the above situation via Mennicke-Newmann lemma, this describes the group structure completely.)

Remark. *The expectation is natural in K-theory!*

The Suslin Matrices $S_r(v, w)$

The construction of the Suslin matrices $S_r(v, w)$ is possible once we have two row vectors v, w. It becomes more interesting if their dot product $vw^t = 1$. (The vectors are then automatically **unimodular vectors**.) A. Suslin's inductive definition: Let

$$v = (a_0, a_1, \ldots, a_r) = (a_0, v_1),$$

with

$$v_1 = (a_1, \ldots, a_r), w = (b_0, b_1, \ldots, b_r) = (b_0, w_1),$$

with $w_1 = (b_1, \ldots, b_r)$.

Set $S_0(v,w) = a_0$, and set

$$S_r(v,w) = \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1,w_1) \\ -S_{r-1}(w_1,v_1)^t & b_0 I_{2^{r-1}} \end{pmatrix}.$$

1. $$\begin{aligned} S_r(v,w)S_r(w,v)^t &= (v.w^t)I_{2^r} \\ &= S_r(w,v)^t S_r(v,w), \end{aligned}$$

2. $\det S_r(v,w) = (v.w^t)^{2^{r-1}}$, for $r \geq 1$.

The Suslin matrices were introduced by A. Suslin to show that a unimodular vector of the form $(a_0, a_1, a_2^2, \ldots, a_r^r)$ could be completed to an invertible matrix.

Example

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & a_1 & 0 & a_2 & a_3 \\ 0 & a_0 & 0 & 0 & 0 & a_1 & -b_3 & b_2 \\ 0 & 0 & a_0 & 0 & -b_2 & a_3 & b_1 & 0 \\ 0 & 0 & 0 & a_0 & -b_3 & -a_2 & 0 & b_1 \\ -b_1 & 0 & a_2 & a_3 & b_0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & b_2 & 0 & b_0 & 0 & 0 \\ -b_2 & a_3 & 0 & -a_1 & 0 & 0 & b_0 & 0 \\ -b_3 & -a_2 & 0 & -a_1 & 0 & 0 & 0 & b_0 \end{pmatrix}$$

$$S_3((a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3))$$

The Special and Elementary Unimodular Vector Groups

We define a new group which promises to throw light on problems regarding completions of unimodular vectors, not necessarily of above type.

Definition. *The Special Unimodular Vector group $SUm_r(R)$ is the subgroup of $Sl_{2^r}(R)$ generated by the Suslin matrices $S_r(v, w)$, as v varies in $Um_{r+1}(R)$, and for some w with $v.w^t = 1$.*

The Elementary Unimodular Vector group $EUm_r(R)$

is the subgroup of $SUm_{r+1}(R)$ *generated by the Suslin matrices* $S_r(e_1\varepsilon, e_1\varepsilon^{t^{-1}})$, *for* $\varepsilon \in E_{r+1}(R)$.

Definition. *For* $1 \leq i \leq r$, $\lambda \in R$,

$$\begin{aligned} E_r(e_i)(\lambda) &= S_r(e_1 + \lambda e_{i+1}, e_1), \\ E_r(e_i^*)(\lambda) &= S_r(e_1, e_1 + \lambda e_{i+1}) \end{aligned}$$

Examples:

$$\begin{pmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda \\ -\lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_2(e_2)(\lambda) \qquad E_2(e_2^*)(\lambda)$$

$$\boldsymbol{E_3(e_3^*)(\lambda)^{bot}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It seems natural to expect that the elementary unimodular generators $\boldsymbol{E(e_i)(\lambda)}$, $\boldsymbol{1 \leq i \leq r+1}$, $\boldsymbol{\lambda \in R}$, should suffices to generate $\boldsymbol{EUm_r(R)}$. However, this is not the case as a simple example will show: Note that elementary generators of type $\boldsymbol{E_2(e_p)(\mu)}$, $\boldsymbol{E_2(e_q)(\nu)}$, are (elementary) symplectic. However, $\boldsymbol{S_2(e_2+e_1, e_2)}$ is not symplectic, and so cannot be written as a product of these generators.

Definition. *For* $\boldsymbol{2 \leq i \leq r+1}$, $\boldsymbol{\lambda \in R}$, *let*

$$\begin{aligned} \boldsymbol{E_r(e_{i1})(\lambda)} &= \boldsymbol{S_r(e_i+\lambda e_1, e_i)}, \\ \boldsymbol{E_r(e_{i1}^*)(\lambda)} &= \boldsymbol{S_r(e_i, e_i+\lambda e_1)}. \end{aligned}$$

Example:

$$\begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{pmatrix}$$

$$E_2(e_{12})(\lambda) \qquad E_2(e_{12}^*)(\lambda)$$

One has the structure theorem for the Elementary Unimodular vector group $EUm_r(R)$:

Theorem. [(Selby Jose and Ravi Rao)] *Let R be a commutative ring in which 2 is invertible. Then $EUm_r(R)$ is generated by elements of the form $E(e_i)(w)$, $E(e_j^*)(x)$, $E(e_{1i})(y)$, $E(e_{1i}^*)(z)$, for $w, x, y, z \in R$, $1 \leq i, j \leq r+1$.*

$\square$

A Witt Group Symbol on the Orbit Space

We shall show that it is possible to define a group structure on the orbit space $Um_n(A)/E_n(A)$, $n \geq 3, d = \dim(A) \leq (2n-3)$, which has a Witt group structure, similar to the one which Vaserstein did when $d = 2$, $n = 3$. However, we need to assume, for simplicity, that -1 is a square in the ring. In particular, under the additional hypothesis, we can recover van der Kallen's theorem, as well as Vaserstein's theorem.

The idea of the proof in a nutshell is to imitate the Vaserstein construction of $W_E(R)$, and to construct a new Witt group $\boldsymbol{W_{EUm}(R)}$, which we call the Elementary Unimodular Witt group. The (equivalence class) of the Suslin matrices will offer the natural access to the orbit space into this Witt group. The fact that the addition of two vectors can be defined will be due to the Mennicke-Newmann lemma. It will be possible to reach the situation described in this lemma due to the analogue of the famous lemma of Vaserstein that elementary orbit equals elementary symplectic orbit.

Theorem. *The Vaserstein-Suslin symbol*

$$S : Um_n(R)/E_n(R) \longrightarrow W_{EUm}(R)$$

$$[v] \longrightarrow [S_n(v, w)] \in W_{EUm}(R)$$

is an isomorphism, for $n \geq (d + 3)/2\}$, d *being the stable dimension of* R*. Moreover, if* $n \geq \max\{3, d/2 + 2\}$*, then this is the universal weak Mennicke symbol.*

Cohn Orbits Versus Elementary Orbits

Definition. *Let*

$$v = (a_0, a_1, \ldots, a_r),$$
$$w = (b_0, b_1, \ldots, b_r)$$

with $v.w^t = 1$*. We say that the vector*

$$v^* = vC_{ij}(\lambda) =$$
$$(a_0, \ldots, a_i + \lambda b_j, \ldots, a_j - \lambda b_i, \ldots, a_r),$$

for $0 \leq i \neq j \leq r$*, is a* $\mathit{Cohn\ transform}$ *of* v *w.r.t. the vector* w*, with* $v.w^t = 1$*.*

The Cohn orbit of a vector $\boldsymbol{v}$ *is the vectors got by a sequence of Cohn transforms of* $\boldsymbol{v}$ *w.r.t. suitable vectors* $\boldsymbol{w}$ *with* $\boldsymbol{v}.\boldsymbol{w}^t = \boldsymbol{1}$ *or to a Cohn transform of* $\boldsymbol{v}$.

We show that the Cohn orbit equals the elementary orbit:

$$\{(a,b,c);(a',b',c')\} \xrightarrow{C_{02}(-\lambda)} \{(a-\lambda c',b,c+\lambda a');(a',b',c')\}$$

$$\xrightarrow{C_{12}(-1)} \{(a-\lambda c',b,c+\lambda a');(a',b'-(c+\lambda a'),c'+b)\}$$

$$\xrightarrow{C_{02}(\lambda)} \{(a-\lambda c'+\lambda(c'+b),b,(c+\lambda a')-\lambda c')\};$$

$$\begin{aligned} & \{(a',b'-(c+\lambda a'),c'+b)\} \\ = \quad & \{(a+\lambda b,b,c);(a',b'-c-\lambda a',c'+b)\} \\ \xrightarrow{C_{12}(1)} \quad & \{(a+\lambda b,b,c);(a',b'-\lambda a',c')\} \end{aligned}$$

Key Lemma

Let $v = (a_0, a_1, \ldots, a_r)$, $w = (b_0, b_1, \ldots, b_r)$, with $vw^t = 1$. Then, for $2 \leq i \leq r+1$, $r \geq 2$,

$$S_r(e_1, e_1 + \lambda e_i)^{top} S_r(v, w) S_r(e_1, e_1 + \lambda e_i)^{bot} = S_r(vE_{i1}(-\lambda), wE_{1i}(\lambda))$$

$$S_r(e_1 + \lambda e_i, e_1)^{bot} S_r(v, w) S_r(e_1 + \lambda e_i, e_1)^{top} = S_r(vE_{1i}(\lambda), wE_{i1}(-\lambda)).$$

(The case when $r = 1$ is left to the reader.)

Moreover, if $1 \leq i \leq r$, then

$$S_r(e_1 + \lambda e_{i+1}, e_1)^{top} S_r(v, w) S_r(e_1 + \lambda e_{i+1}, e_1)^{bot} = S_r(vC_{0i}(-\lambda), w)$$

$$S_r(e_1, e_1 + \lambda e_{i+1})^{bot} S_r(v, w) S_r(e_1, e_1 + \lambda e_{i+1})^{top} = S_r(v, wC_{0i}(-\lambda))$$

Key Lemma is an Action

Lemma. *Let*

$$\widetilde{J}_r = \begin{cases} J_r & \text{if } r \text{ even} \\ \begin{pmatrix} 0 & J_{r-1} \\ J_{r-1} & 0 \end{pmatrix} & \text{if } r \text{ odd.} \end{cases}$$

Then for $c = e_i, e_i^*, 1 \le i \le r, \lambda \in R$,

$$\widetilde{J}_r E(c)(\lambda)^{tb^T} \widetilde{J}_r^{-1} = \begin{cases} E(c)(\lambda)^{bt} & \text{if } r \text{ even} \\ E(c)(\lambda)^{tb} & \text{if } r \text{ odd.} \end{cases}$$

Corollary. *If r is even, $EUm_r(R)^{tb}$ acts on*

$$S = \{S_r(v,w) | S_r(v,w) \in SUm_r(R)\}.$$

For any r, $EUm_r(R)$ acts on S.

Proof. For $\alpha \in EUm_r(R)^{tb}$ (respectively $EUm_r(R)$), the action is given by $\alpha S_r(v,w)(\widetilde{J}_r \alpha^T \widetilde{J}_r^{-1})$. □

Remark. *1. It is not too difficult to show even when r is odd that $EUm_r(R)^{tb}$ acts on S.*

2. Since $EUm_r(R) \subset EUm_r(R)^{tb}$. Hence, via the above remark, $EUm_r(R)$ acts on the set S as above

Notation. *We will write* α^* *for* $\widetilde{J}_r\alpha^t\widetilde{J}_r^{-1}$*, when* r *is even, and the appropriate matrix when* r *is odd to ensure that one gets an action on* S*.*

Examples:

For $0 < i \neq j \leq r+1$, we have for $\lambda = -2xy$. If

$$\alpha = \{[E(e_i)(x), E(e_j)(y)]\},$$

then $\alpha^* = \alpha^{-1}$, and

$$S_r(vC_{ij}(\lambda), w) = \alpha S_r(v, w)\alpha^{-1}$$

If

$$\beta = \{[E(e_i^*)(x), E(e_j^*)(y)]\},$$

then $\beta^* = \beta^{-1}$, and

$$S_r(v, wC_{ij}(\lambda)) = \beta S_r(v, w)\beta^{-1}$$

If

$$\gamma = \{[E(e_{j-1})(x), E(e_{i-1}^*)(y)]\},$$

then $\gamma^* = \gamma^{-1}$, and

$$S_r(vE_{ij}(\lambda), wE_{ji}(-\lambda)) = \gamma S_r(v, w)\gamma^{-1}$$

Central Positioning

Definition. *Let* $\alpha, \beta \in Gl_{2r}(R)$. *We shall think of them as* 4×4 *block matrices*

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix}$$

Define

$$\alpha \odot \beta = \begin{pmatrix} \alpha_{11} & 0 & 0 & \alpha_{12} \\ 0 & \beta_{11} & \beta_{12} & 0 \\ 0 & \beta_{21} & \beta_{22} & 0 \\ \alpha_{21} & 0 & 0 & \alpha_{22} \end{pmatrix} \in Gl_{4r}(R)$$

We can loosely say that β *is in the center of* $\alpha \odot \beta$. *For example, note that the Suslin matrix* $S_r(v, w)$ *has a* $S_{r-1}(v^*, w^*)$ *in its center. (Of course, the Suslin matrix* $S_r(v, w)$ *is not exactly got by centering.)*

$$\begin{pmatrix} a_0 & 0 & a_1 & a_2 \\ 0 & a_0 & -b_2 & b_1 \\ -b_1 & a_2 & b_0 & 0 \\ -b_2 & -a_1 & 0 & b_0 \end{pmatrix}$$

Equivalence Relation on the Suslin Matrices

We shall always regard $\alpha \in Sl_{2^r}(R)$ to be sitting inside $Sl_{2^{r+1}}(R)$ via centering, i.e. α should be replaced by $(I \odot \alpha)$.

Given

$$S_r(v, w), S_r(v^*, w^*),$$

we say that they are equivalent

$$S_r(v, w) \simeq S_r(v^*, w^*)$$

if there exist a telescope of elementary unimodular matrices

$$\boldsymbol{\varepsilon_1 \in EUm_r(R), \varepsilon_2 \in EUm_{r+1}(R),}$$

and elementary matrices

$$\boldsymbol{\varepsilon, \varepsilon^* \in E_{r+1}(R)}$$

such that

$$\boldsymbol{(\varepsilon_2\varepsilon_1)(S_r(e_1\varepsilon, e_1\varepsilon^{t^{-1}}) \odot S_r(v, w))(\varepsilon_2\varepsilon_1)^* = S_r(e_1\varepsilon^*, e_1\varepsilon^* t^{-1}) \odot S_r(v^*, w^*))}$$

Remark. *One can have a more general definition in which the telescope will have more matrices. This is needed in some applications.*

It can be verified that $\simeq$ is an equivalence relation on the set of all Suslin matrices of type r. Transitivity is non-trivial, and needs the following type of observation

An Analogue of Vaserstein Lemma

Given a Suslin matrix $S_r(v,w)$, and a $\theta \in E_{r+1}(R)$, there exists an elementary unimodular matrix $\varepsilon \in EUm_r(R)^{tb}$ such that

$$\varepsilon S_r(v,w)\varepsilon^* = S_r(v\theta, w\theta^{t^{-1}}),$$

with $\varepsilon\varepsilon^* = I$.

This lemma may be regarded as the equivalent statement to the famous Vaserstein lemma which states that

$$e_1 E_{2n}(A) = e_1 ESp_{2n}(A).$$

Injectivity of the Vaserstein-Suslin Symbol

If one has $I =$

$$(\varepsilon_2\varepsilon_1)(S_r(e_1\varepsilon, e_1\varepsilon^{t^{-1}}) \odot S_r(v, w))(\varepsilon_1\varepsilon_2)^*.$$

Then multiplying by ${\varepsilon_2}^{-1}$ gives

$$(S_r(e_1\varepsilon, e_1\varepsilon^{t^{-1}}) \odot S_r(\widetilde{v}, \widetilde{w})) = S_{r+1}(e_1\theta, e_1\theta^{t^{-1}}),$$

for some elementary θ. (Here $\widetilde{v}$ is in the elementary orbit of v, etc.)

Note that the left hand side is a matrix got by centering two matrices, and hence so is the right hand side.

Moreover, a Suslin matrix which is got by centering is essentially of the form

$$S_r(x, y) \odot S_r(\widetilde{x}, \widetilde{y})^{-1},$$

where $\widetilde{x}$ is in the elementary orbit of x, etc. Since $S_r(x, y) \in EUm_r(R)$, the result follows.

The Elementary Unimodular Witt Group $W_{EUm}(R)$

One can show that $\odot$ defines an addition on the equivalence classes of Suslin matrices of size 2^r; and this is actually an abelian group $W_{EUm}(R)$ — known as the *Elementary Unimodular Witt Group.*

Once we establish the closure of the addition operation, it is easy to show that $\odot$ is associative. We have I as the identity element, and

$$[S_r(v,w)]^{-1} = [S_r(v,w)^{-1}] = [S_r(w,v)^t].$$

It is also easy to show that $\odot$ is commutative.

Why is our group and the van der Kallen group structure isomorphic? The reason is that we can show that the product $*$ in both the groups is the obvious expected K-theoretic product:

$$(a_0(1 - a_0), a_1, \ldots , a_n)] = \\ [(a_0, a_1, \ldots , a_n)] * [[(1 - a_0), a_1, \ldots , a_n)]$$

Addition is by central placement. The surjectivity of the Suslin symbol will follow if one can manipulate with the 'sum' of two Suslin matrices by means of Elementary unimodular matrices and succeed in getting a single Suslin matrix in the center.

$$\begin{pmatrix} \boldsymbol{a_0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{a'_2} & \boldsymbol{-a_3} \\ \mathbf{0} & \boldsymbol{a_0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{b_3} & \boldsymbol{b'_2} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{a_0} & \mathbf{0} & \boldsymbol{-a_2} & \boldsymbol{-a_3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{a_0} & \boldsymbol{b_3} & \boldsymbol{-b_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{b_2} & \boldsymbol{-a_3} & \boldsymbol{b_0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{b_3} & \boldsymbol{a_2} & \mathbf{0} & \boldsymbol{b_0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{-b'_2} & \boldsymbol{-a_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{b_0} & \mathbf{0} \\ \boldsymbol{b_3} & \boldsymbol{-a'_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{b_0} \end{pmatrix}$$

$$S_2[a_0, -a_2, -a_3; b_0, -b_2, -b_3] \odot S_2[a_0, a'_2, -a_3; b_0, b'_2, -b_3]$$

Let us call it α

Remark. *It is due to the Mennicke-Newmann Lemma that we are able to consider starting with the above matrix. But we have to ensure first that we can reduce to this case. This is done via the analogue of the lemma of Vaserstein that*

$$vE_{2n}(A) = vEsp_{2n}(\phi),$$

for any alternating matrix $\phi \in Sl_{2n}(A)$ *of Pffafian one.*

An Analogue of Vaserstein Lemma

Given two Suslin matrces $S_r(v, w)$, $S_r(v^*, w^*)$, with $v \notin v^* E_{r+1}(R)$, and a $\theta \in E_{r+1}(R)$, there exists an elementary unimodular matrix $\varepsilon \in EUm_r(R)^{tb}$ such that

$$\varepsilon S_r(v, w)(\varepsilon)^* = S_r(v\theta, w\theta^{t^{-1}}),$$

with $\varepsilon S_r(v^*, w^*)(\varepsilon)^* = S_r(v^*, w^*)$.

Remark. *Our proof of this needs an application of the Local Global Principle for* $EUm_r(R[X])$ *to reduce it to the previous version of the lemma when* $v^* = e_1 = w^*$.

$$
\begin{aligned}
\alpha_1 &= \varepsilon_1 \alpha (\varepsilon_1)^* \\
\varepsilon_1 &= [E_3(e_2)(1)^{top}, E_3(e_3^*)(-1)^{bot}] \times \\
&\quad [E_3(e_2^*)(1)^{top}, E_3(e_1)(-1)^{bot}] \\
\alpha_2 &= \varepsilon_2 \alpha_1 (\varepsilon_2)^* \\
\varepsilon_2 &= [E_3(e_2^*)(a_2)^{top}, E_3(e_1^*)(-1)^{bot}] \\
\alpha_3 &= \varepsilon_3 \alpha_2 (\varepsilon_3)^* \\
\varepsilon_3 &= [E_3(e_2)(a_2')^{top}, E_3(e_1^*)(-1)^{bot}] \\
\alpha_4 &= \varepsilon_4 \alpha_3 (\varepsilon_4)^* \\
\varepsilon_4 &= [E_3(e_3)(a_3)^{top}, E_3(e_1^*)(1)^{bot}]
\end{aligned}
$$

$$
\begin{aligned}
\alpha_5 &= \varepsilon_5 \alpha_4 (\varepsilon_5)^* \\
\varepsilon_5 &= E_3(e_1^*)(-b_0)^{bot} \\
\alpha_6 &= \varepsilon_6 \alpha_5 (\varepsilon_6)^* \\
\varepsilon_6 &= E_3(e_3^*)(b_3)^{top}, E_3(e_1^*)(1)^{bot}] \\
\alpha_7 &= \varepsilon_7 \alpha_6 (\varepsilon_7)^* \\
\varepsilon_7 &= [E_3(e_2^*)(1)^{top}, E_3(e_1)(-1)^{bot}] \times \\
&\quad [E_3(e_2)(1)^{top}, E_3(e_1^*)(-1)^{bot}] \times \\
&\quad [E_3(e_2^*)(1)^{top}, E_3(e_1)(-1)^{bot}] \\
\alpha_8 &= \varepsilon_8 \alpha_1 (\varepsilon_8)^* \\
\varepsilon_8 &= E_3(e_2^*)(a_0)^{top}
\end{aligned}
$$

Unimodular $2 \times n$-Vectors

We briefly discuss a second application — we still have to work out the details for it. Does the orbit set of unimodular $2 \times n$ matrices (i.e. right invertible $2 \times n$ matrices)

$$Um_{2,n}(R)/E_n(R)$$

have a group structure? Size restrictions are due to:

Mennicke Newmann Lemma for Unimodular $2 \times n$-Vectors

Let $n \geq 4$, and $d \leq 2n-5$. Let two elements of $Um_{2,n}(R)/E_n(R)$ be given. Then we may choose representatives of the form

$$\begin{pmatrix} a & b & y_{11} & y_{12} & z_{11} & \cdots & z_{1n-4} \\ g & -a & y_{21} & y_{22} & z_{21} & \cdots & z_{2n-4} \end{pmatrix},$$

$$\begin{pmatrix} 1-a & -b & y_{11} & y_{12} & z_{11} & \cdots & z_{1n-4} \\ -g & 1+a & y_{21} & y_{22} & z_{21} & \cdots & z_{2n-4} \end{pmatrix},$$

respectively.

Remark. *It is easy to see that if*

$$X = \begin{pmatrix} a & b \\ g & -a \end{pmatrix}, Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix},$$

$$Z = \left(z_{11} \cdots z_{1n-4} \; z_{21} \cdots z_{2n-4} \right),$$

then $X(I - X, Y, Z)$ *is a unimodular* $2 \times n$ *matrix. Naturally, we expect it to be the product of the above two!*

There is a very natural way to approach this problem based on the earlier theory.

Let $(v_1, w_1), (v_2, w_2), (v_3, w_3), (v_4, w_4)$ be four pairs of vectors with $v_i.w_i^t = 1$, for all $1 \leq i \leq 4.$, and with $\forall i$,

$$\begin{pmatrix} v_i \\ w_i \end{pmatrix} \in Um_{2n}(A).$$

Consider the Suslin matrix $S_r(v, w)$ corresponding to the 'cocatenated vectors'

$$v = (v_1, v_2, v_3, v_4), w = (w_1, w_2, w_3, w_4)$$

Of course, $S_r(v, w)$ is elementary unimodular, so we cannot hope to get much if we study its class in $W_{EUm}(R)$.

However, we can study its class in a modified Witt group. The fact that we are dealing with orbit space of $2 \times n$ matrices by $E_n(R)$ places a natural restriction on us. This allows us to define a natural sequence of subgroups $\widetilde{EUm}_n(R)$ of $EUm_n(R)$, $\forall\ n$. We shall study the class of $S_r(v, w)$ w.r.t this subgroup.

The injectivity of the natural map

$$Um_{2n}(R)/E_n(R) \longrightarrow W_{\widetilde{EUm}}(R)$$

is not difficult to show.

The surjectivity part is the key point. The above version of the Mennicke-Newmann lemma should allow us to conclude that the class of the $\odot$ sum can be cut down in size, as before. However, the sizes of what we have to deal with is just too large to physically work out - the minimum size is $\mathbf{2^{16}}$. So we have to work it out more efficiently. The combinatorial aspect of the **Key Lemma** gives a way out, and there is a method in which we have to work out that the class of the $\odot$ sum can be cut down working with $\mathbf{16 \times 16}$ matrices essentially.

Completing Unimodular Polynomial Vectors of Size $\geq \frac{d}{2} + 2$ over $R[X]$

One expects to prove a M. Karoubi type theorem for the Elementary Unimodular Witt group, viz.

$$W_{EUm}(R[X]) \simeq W_{EUm}(R), \text{ if } \frac{1}{2} \in R.$$

Moreover, as is known for the Elementary symplectic Witt group, one expects that the Elementary Unimodular Witt group is also k-divisible if $\frac{1}{k} \in R$.

Hence, one hopes to derive from this that

$$Um_r(R[X] = e_1 Sl_r(R[X]) \text{ if } \frac{1}{r!} \in R,$$

for $r \geq \{\frac{d}{2} + 2\}$, as was shown by M. Roitman in positive characteristics.

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & -1 & 0 & a_2' & -a_3 \\ 0 & a_0 & 0 & 0 & 0 & -\lambda & b_3 & b_2' \\ 0 & 0 & a_0 & 0 & -a_2 & -a_3 & 0 & 0 \\ 0 & 0 & 0 & a_0 & b_3 & -b_2 & 0 & 0 \\ 0 & 0 & b_2 & -a_3 & b_0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & a_2 & 0 & b_0 & 0 & 0 \\ -b_2' & -a_3 & \lambda & 0 & 0 & 0 & b_0 & 0 \\ b_3 & -a_2' & 0 & 1 & 0 & 0 & 0 & b_0 \end{pmatrix}$$

$$\alpha_1 = \varepsilon_1 \alpha (\varepsilon_1)^*$$

$$\varepsilon_1 = [E_3(e_2)(1)^t, E_3(e_3^*)(-1)^b][E_3(e_2^*)(1)^t, E_3(e_1)(-1)^b]$$

^{t} denotes ^{top}, and ^{b} denotes ^{bot} above.

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & -1 & 0 & a_2' & -a_3 \\ 0 & a_0 & 0 & 0 & 0 & -\lambda & b_3 & 0 \\ 0 & 0 & a_0 & 0 & 0 & -a_3 & -a_2a_2' & 0 \\ 0 & 0 & 0 & a_0 & b_3 & -b_2 & 0 & -a_2b_2 \\ a_2b_2 & 0 & b_2 & -a_3 & b_0 & 0 & 0 & 0 \\ 0 & a_2a_2' & b_3 & 0 & 0 & b_0 & 0 & 0 \\ 0 & -a_3 & \lambda & 0 & 0 & 0 & b_0 & 0 \\ b_3 & -a_2' & 0 & 1 & 0 & 0 & 0 & b_0 \end{pmatrix}$$

$$\alpha_2 = \varepsilon_2 \alpha_1 (\varepsilon_2)^*$$
$$\varepsilon_2 = [E_3(e_2^*)(a_2)^{top}, E_3(e_1^*)(-1)^{bot}]$$

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & -1 & 0 & 0 & -a_3 \\ 0 & a_0 & 0 & 0 & 0 & -\lambda & b_3 & 0 \\ 0 & 0 & a_0 & 0 & 0 & -a_3 & -a_2a_2' & 0 \\ 0 & 0 & 0 & a_0 & b_3 & 0 & 0 & -a_2b_2 \\ a_2b_2 & 0 & 0 & -a_3 & b_0 & 0 & 0 & 0 \\ 0 & a_2a_2' & b_3 & 0 & 0 & b_0 & 0 & 0 \\ 0 & -a_3 & \lambda & 0 & 0 & 0 & b_0 & 0 \\ b_3 & 0 & 0 & 1 & 0 & 0 & 0 & b_0 \end{pmatrix}$$

$$\alpha_3 = \varepsilon_3\alpha_2(\varepsilon_3)^*$$

$$\varepsilon_3 = [E_3(e_2)(a_2')^{top}, E_3(e_1^*)(-1)^{bot}]$$

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 & 0 & -\lambda & b_3 & 0 \\ 0 & 0 & a_0 & 0 & 0 & \lambda' a_3 & x & 0 \\ 0 & 0 & 0 & a_0 & b_3 & 0 & 0 & -x' \\ x' & 0 & 0 & 0 & b_0 & 0 & 0 & 0 \\ 0 & -x & b_3 & 0 & 0 & b_0 & 0 & 0 \\ 0 & -\lambda' a_3 & \lambda & 0 & 0 & 0 & b_0 & 0 \\ b_3 & 0 & 0 & 1 & 0 & 0 & 0 & b_0 \end{pmatrix}$$

$$\lambda' = (\lambda - 1); x = -a_2 a_2' - a_3 b_3, x' = (1 - a_0 b_0)$$

$$\alpha_4 = \varepsilon_4 \alpha_3 (\varepsilon_4)^*$$

$$\varepsilon_4 = [E_3(e_3)(a_3)^{top}, E_3(e_1^*)(1)^{bot}]$$

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 & 0 & -\lambda & b_3 & 0 \\ 0 & 0 & a_0 & 0 & 0 & \lambda' a_3 & y & 0 \\ 0 & 0 & 0 & a_0 & b_3 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & b_3 & 0 & 0 & -\lambda' b_0 & 0 & 0 \\ 0 & -\lambda' a_3 & \lambda & 0 & 0 & 0 & -\lambda' b_0 & 0 \\ b_3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$y = x - a_0 b_0$$

$$\alpha_5 = \varepsilon_5 \alpha_4 (\varepsilon_5)^*$$

$$\varepsilon_5 = E_3(e_1^*)(-b_0)^{bot}$$

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & a_0 & 0 & 0 & 0 & -\lambda & -\lambda' b_3 & 0 \\ 0 & 0 & a_0 & 0 & 0 & \lambda' a_3 & y^* & 0 \\ 0 & 0 & 0 & a_0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y^* & -\lambda' b_3 & 0 & 0 & -\lambda' b_0 & 0 & 0 \\ 0 & \lambda' a_3 & \lambda & 0 & 0 & 0 & -\lambda' b_0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_6 = \varepsilon_6 \alpha_5 (\varepsilon_6)^*$$
$$\varepsilon_6 = E_3(e_3^*)(b_3)^{top}, E_3(e_1^*)(1)^{bot}]$$

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & a_0 & 0 & a_0 & 0 & 0 & 0 & 1 \\ 0 & 0 & a_0 & 0 & y^* & \lambda' a_3 & 0 & 0 \\ 0 & 0 & 0 & a_0 & -\lambda' b_3 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & \lambda' a_3 & -\lambda' b_0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda' b_3 & -y^* & 0 & -\lambda^* b_0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_7 = \varepsilon_7 \alpha_6 (\varepsilon_7)^*$$

$$\varepsilon_7 = [E_3(e_2^*)(1)^{top}, E_3(e_1)(-1)^{bot}]$$

$$[E_3(e_2)(1)^t, E_3(e_1^*)(-1)^b][E_3(e_2^*)(1)^t, E_3(e_1)(-1)^b]$$

^{t} denotes ^{top} and ^{b} denotes ^{bot} above.