Relative Rates of Growth

Many of the common functions tend to take bigger and bigger values when the argument is getting bigger. In other words, for many common functions $f(x)$ we have $\lim_{x\to+\infty} f(x) = +\infty$. Sometimes this limit condition is all we need. But often a more detailed analysis of the behavior of $f(x)$ as $x \to +\infty$ is desired.

In this section we will discuss how to compare the rates at which different functions of x grow as x becomes large. *We will consider only the functions whose values eventually become positive as* $x \to +\infty$.

To illustrate this, consider the following two functions.

$$f(x) = x^{10}$$

$$g(x) = 10^x$$

Both of these functions are increasing to $+\infty$ as $x \to +\infty$. However $g(x)$ increases much faster, for big x. Indeed, for $x = 100$ we have

$$f(100) = 100^{10} = 10^{20} = 100000000000000000000 \ \ \text{(21 digits)},$$

while

$$g(100) = 10^{100} = 10000...000 \ \ \text{(101 digits)}.$$

The comparison is even more striking when $x = 1000$.

$$f(1000) = 1000^{10} = 10^{30} = 1000000000000000000000000000000 \ \text{(31 digits)},$$

while

$$g(1000) = 10^{1000} = 10000...000 \ \ \text{(1001 digits)!}$$

Clearly, $g(x)$ grows much faster than $f(x)$ as $x \to +\infty$.

The comparison of these functions can be made precise if we use the following definition.

Definition 1. Let $f(x)$ and $g(x)$ be positive for x sufficiently large.
We say that $f(x)$ **grows faster than** $g(x)$ as $x \to +\infty$, if

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = +\infty$$

or, equivalently, if

$$\lim_{x\to+\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that $g(x)$ **grows slower than** $f(x)$ as $x \to +\infty$. We will denote this by $g(x) << f(x)$.

Definition 2. Let $f(x)$ and $g(x)$ be positive for x sufficiently large. We say that $f(x)$ and $g(x)$ **grow at the same rate** as $x \to +\infty$, if

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = M \neq 0. \qquad (M \text{ is a finite non-zero number})$$

We will denote this by $g(x) \asymp f(x)$.

Example 1. x grows faster than $\ln x$ as $x \to +\infty$, since

$$\lim_{x\to+\infty} \frac{x}{\ln x} = \lim_{x\to+\infty} \frac{1}{(1/x)} = +\infty.$$

Example 2. e^x grows faster than x as $x \to +\infty$, since

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0.$$

Example 3. e^x grows faster than 2^x as $x \to +\infty$, since

$$\lim_{x\to+\infty} \frac{2^x}{e^x} = \lim_{x\to+\infty} \left(\frac{2}{e}\right)^x = 0,$$

because $\frac{2}{e} < 1$ and $\lim\limits_{x\to+\infty} a^x = 0$ for $0 < a < 1$.

Example 4. $3x^2$ and $x^2 + 5$ grow at the same rate as $x \to +\infty$, since

$$\lim_{x\to+\infty} \frac{3x^2}{x^2+5} = \lim_{x\to+\infty} \frac{3}{1+\frac{5}{x^2}} = \frac{3}{1+0} = 3 \neq 0.$$

Example 5. $\log_a x$ and $\log_b x$ grow at the same rate as $x \to +\infty$, since

$$\lim_{x\to+\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to+\infty} \frac{(\ln x/\ln a)}{(\ln x/\ln b)} = \frac{\ln b}{\ln a} \neq 0.$$

The following two theorems are useful for analyzing the rates of growth of functions represented by algebraic expressions.

Theorem 1. For any $k > 0$ $f(x) \asymp kf(x)$ as $x \to +\infty$.
Proof.

$$\lim_{x\to+\infty} \frac{kf(x)}{f(x)} = k \neq 0.$$

Theorem 2. If $g(x) << f(x)$ then $f(x) \asymp f(x) \pm g(x)$ as $x \to +\infty$.

Proof.

$$\lim_{x\to+\infty}\frac{f(x)\pm g(x)}{f(x)}=\lim_{x\to+\infty}\frac{1\pm\frac{g(x)}{f(x)}}{1}=\frac{1\pm 0}{1}=1\neq 0.$$

Example 6. $e^x-8x^{10}+15\ln x\asymp e^x$ as $x\to+\infty$, since $8x^{10}\asymp x^{10}$, $15\ln x\asymp\ln x$ and $\ln x<<x^{10}<<e^x$ as $x\to+\infty$.

Example 7. Show that $\sqrt{x^4-2x+5\ln x}$ and $3x^2-\log_2 x$ grow at the same rate as $x\to+\infty$.

Solution 1.

$$\lim_{x\to+\infty}\frac{\sqrt{x^4-2x+5\ln x}}{3x^2-\log_2 x}=\lim_{x\to+\infty}\frac{\sqrt{x^4-2x+5\ln x}\cdot\frac{1}{x^2}}{(3x^2-\log_2 x)\cdot\frac{1}{x^2}}=$$

$$=\lim_{x\to+\infty}\frac{\sqrt{\frac{x^4-2x+5\ln x}{x^4}}}{3-\frac{\log_2 x}{x^2}}=\lim_{x\to+\infty}\frac{\sqrt{1-\frac{2}{x^3}+5\frac{\ln x}{x^4}}}{3-\frac{\ln x}{x^2\ln 2}}=$$

$$=\frac{\sqrt{1-2\cdot 0+5\cdot 0}}{3-0}=\frac{1}{3}\neq 0.$$

Solution 2. By Theorem 1, $2x$ and x grow at the same rate, that is $2x\asymp x$. Also, $5\ln x$ and $\ln x$ grow at the same rate. Then, by Theorem 2, $x^4-2x+5\ln x$ and x^4 grow at the same rate, because $\ln x<<x<<x^4$. Thus, $\sqrt{x^4-2x+5\ln x}$ grows at the same rate as $\sqrt{x^4}=x^2$ as $x\to+\infty$.

Now, again, Theorem 1 implies that $\log_2 x=\frac{\ln x}{\ln 2}$ and $\ln x$ grow at the same rate. By Theorem 2, $3x^2-\log_2 x$ and $3x^2$, grow at the same rate, since $\ln x<<3x^2$.

Thus, $\sqrt{x^4-2x+5\ln x}\asymp x^2\asymp 3x^2\asymp 3x^2-\log_2 x$.

The chain below lists some common functions according to their rate of growth as $x\to+\infty$.

$$\ldots<<(\ln(\ln x))^\alpha<<(\ln x)^\beta<<x^n<<x^m<<a^x<<b^x<<x^x<<\ldots$$
$$\alpha>0\qquad\beta>0\qquad n<m\qquad 1<a<b$$

Exercises

In problems **1 - 18** for each pair of functions given, make a guess about which function grows faster as $x \to +\infty$. Then use a limit to prove your guess. The goal is to be able to determine most relative growth rates without needing to use limits.

1. x^3 and $x^2 + x + 1$

2. x^3 and $x^2 + e^x$

3. $\ln x$ and $\ln 2x$

4. $\ln x$ and $\log_2 x$

5. $\ln x$ and $\log_2(x^2)$

6. $\ln x$ and $(\log_2 x)^2$

7. e^x and e^{2x}

8. e^x and e^{x-1}

9. x and $\sqrt{x}$

10. $\ln x$ and $\sqrt{\ln x}$

11. e^x and $e^{\cos x}$

12. e^x and π^x

13. 2^x and 2^{-x}

14. 2^x and $\log_2 x$

15. $x \ln x - x$ and $x \ln x$

16. $(\ln 2)^x$ and $(\ln 3)^x$

17. 100^x and e^x

18. x^{100} and e^x

19. Use the Definition 1 to prove the following chain of asymptotic inequalities:

$$(\ln(\ln x))^{10} << (\ln x)^7 << x^{1/3} << x^{100} << 2^x << 5^x << x^x$$

20. Consider the function $f(x) = \frac{e^x}{x^{100}}$, for $x > 0$.

a) Find $\lim\limits_{x \to +\infty} f(x)$.

b) Use a graphing calculator to sketch the graph of $f(x)$. Does it match findings in part a)?

c) Find $f'(x)$ and all local maxima and minima of $f(x)$. Use this information to explain the results of part b).

In problems **21 - 28** determine which of the following two functions grows faster, or show that they grow at the same rate.

21. 3^x and $\frac{x+4^x}{2^x+\sin x}$

22. $\sqrt{2x^4 - \ln x}$ and $(3x + \cos^3 x)^2$

23. e^x and $x^5 \cdot 2^x$

24. $e^{2\ln x}$ and x^2

25. $\ln(e^x + \cos x)$ and $x \ln x$

26. $\ln(x^3 + 1)$ and $\ln(x^2 + 100x)$

27. $x^2 + \sin(e^x)$ and $\sin(x^2) + e^x$

28. $e^{x+\sqrt{x}}$ and $e^{x-\sqrt{x}}$

In problems **29 - 32** determine which of the following two functions grows faster, or show that they grow at the same rate. These problems are of higher difficulty.

29.* $e^{\sqrt{\ln x}}$ and x^3

30.* $x^{\ln x}$ and $x^{\ln(x+\sin x)}$

31.* $(\ln x)^{\ln x}$ and x^2

32.* e^x and $(\ln x)^{\sqrt{x}}$