MATH 251

Final Examination

May 3, 2017

FORM A

Name:________________

Student Number:________________

Section:________________

This exam has 16 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE AND ALL OTHER MOBILE DEVICES.

Do not write in this box.

1 through 10:		(60)
11:		(12)
12:		(16)
13:		(17)
14:		(16)
15:		(13)
16:		(16)
Total:		

1. D
2. A
3. B
4. C
5. A
6. B
7. C
8. B
9. A
10. B

11. (a) F
(b) T
(c) T
(d) T
(e) F
(f) T

1. (6 points) Consider the autonomous equation

$$y' = y^2(25 - y^2).$$

Given the initial conditions $y(2017) = y_0$, find all possible values of y_0 such that $\lim_{t\to\infty} y(t) = 0$.

(a) $y_0 = 0$

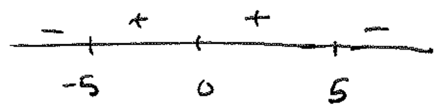
(b) $-5 < y_0 < 5$

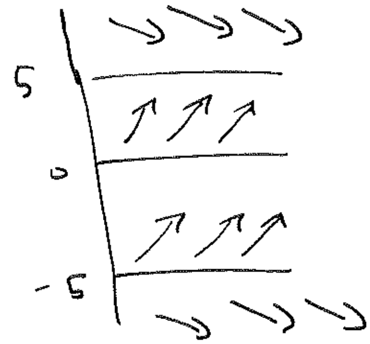
(c) $-5 < y_0$

(d) $-5 < y_0 \le 0$

$y^2(25-y^2)$

$= y^2(5-y)(5+y)$





2. (6 points) Consider the initial/boundary value problems below. Which is certain to have a unique solution for every value of α?

I $(t^2 - 9)y'' + 4ty = 0, \quad y(\alpha) = 0, \quad y'(\alpha) = \alpha.$

II $y'' + 6y = 0, \quad y(0) = 0, \quad y'(\alpha) = 0.$

(a) Neither I nor II.

(b) I only.

(c) II only.

(d) Both I and II.

I loses existence/uniqueness at $\alpha = \pm 3$,

II loses uniqueness at $\alpha = \frac{(2n-1)\pi n}{2\sqrt{6}}$, $n \in \mathbb{Z}$

3. (6 points) Which of the equations below is an exact equation whose solution is the function

$$2x^2y^3 + \cos(x)\sin(y) = C\,?$$

(a) $\big(6x^2y^2 + \sin(x)\sin(y)\big)y' + 4xy^3 - \cos(x)\cos(y) = 0$

(b) $\big(6x^2y^2 + \cos(x)\cos(y)\big)y' + 4xy^3 - \sin(x)\sin(y) = 0$

(c) $\big(4xy^3 - \cos(x)\cos(y)\big)y' + 6x^2y^2 + \sin(x)\sin(y) = 0$

(d) $\big(4xy^3 - \sin(x)\sin(y)\big)y' + 6x^2y^2 + \cos(x)\cos(y) = 0$

$$\frac{\partial \Phi}{\partial y} = 6x^2y^2 + \cos(x)\cos(y) \quad , \quad \frac{\partial \Phi}{\partial x} = 4xy^3 - \sin(x)\sin(y)$$

4. (6 points) Which of the equations below has solutions that converge to a finite but **nonzero** limit as $t \to \infty$?

(a) $y'' - 2y' + y = 0$

(b) $y'' + 3y' + 2y = t$

(c) $y'' + 2y' + 5y = 2 + e^{-t}$

(d) $y'' + 9y = 0$

$$y = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t) + \frac{2}{5} + \frac{1}{4}e^{-t}$$

$$\to \frac{2}{5} \text{ as } t \to \infty.$$

5. (6 points) Suppose $y_1(t)$ and $y_2(t)$ are any two solutions of the second order linear equation

$$(1+t^2)y'' + 4t\,y' + (1-t)y = 0.$$

Which function below can possibly be their Wronskian, $W(y_1, y_2)(t)$?

(a) $W(y_1, y_2)(t) = \dfrac{-5}{(1+t^2)^2}$

(b) $W(y_1, y_2)(t) = -4\arctan(x)$

(c) $W(y_1, y_2)(t) = 7e^{-2t^2}$

(d) $W(y_1, y_2)(t) = 2(1+t^2)^2$

as $W = \frac{c}{(1+t^2)^2}$

$p(t) = \frac{4t}{1+t^2}$, $W = Ce^{-\int p\,dt}$
$= Ce^{-2\ln(1+t^2)}$

6. (6 points) Find the Laplace transform $\mathcal{L}\{u_6(t)te^{2t-12}\}$.

(a) $F(s) = e^{-6s-12}\,\dfrac{1}{(s-2)^2}$

(b) $F(s) = e^{-6s}\,\dfrac{6s-11}{(s-2)^2}$

(c) $F(s) = e^{-6s-12}\,\dfrac{6s+13}{(s-2)^2}$

(d) $F(s) = e^{-6s}\,\dfrac{1}{(s-2)^2}$

$= e^{-6s}\mathcal{L}\{(t+6)e^{2t}\}$

$= e^{-6s}\left(\frac{6}{s-2} + \frac{1}{(s-2)^2}\right)$

$= e^{-6s}\,\frac{6s-11}{(s-2)^2}$

7. (6 points) Consider the following nonlinear system of equations:

$$\begin{aligned} x' &= xy + 4y \\ y' &= xy + x \end{aligned} .$$

Which of the statements below is FALSE?

(0,0) saddle

~~[illegible]~~ (-4, -1) nodal sink

(a) The system has exactly 2 critical points.

(b) One of its critical points is asymptotically stable.

(c) It has at least 2 unstable critical points.

(d) One of its critical points is a saddle point.

8. (6 points) Consider the two linear partial differential equations.

$$\begin{array}{ll} \textbf{I} & e^t u_{xx} - u_{xt} + 5x^3 u_t = 0 \\ \textbf{II} & u_{xx} + 6u_{tx} - 3u_{tt} = 0 \end{array}$$

Use the substitution $u(x,t) = X(x)T(t)$, where $u(x,t)$ is not the trivial solution, and attempt to separate each equation into two ordinary differential equations. Which statement below is true?

(a) Neither equation is separable.

(b) Only I is separable.

(c) Only II is separable.

(d) Both equations are separable.

$I: \; e^t X''T - X'T' + 5x^3 XT' = 0$

$\Rightarrow \frac{T'}{T} e^{-t} = \frac{X''}{X' - 5x^3 X}$ (separable)

$II: \; X''T + 6X'T' - 3XT'' = 0$ (not separable)

9. (6 points) Find the steady-state solution, $v(x)$, of the heat conduction problem with nonhomogeneous boundary conditions:

$$4u_{xx} = u_t, \qquad 0 < x < 2, \quad t > 0$$
$$u(0,t) + u_x(0,t) = 5, \qquad u(2,t) = 6$$
$$u(x,0) = 2x + 3$$

(a) $v(x) = x + 4$

(b) $v(x) = 4x - 2$

(c) $v(x) = 2x + 3$

(d) $v(x) = \frac{1}{2}x + 5$

$V(x) = Ax + B$

$V(0) + V'(0) = 5 \Rightarrow A + B = 5$

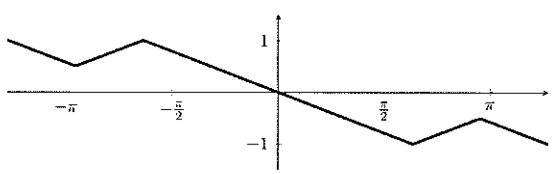
$V(2) = 6 \qquad 2A + B = 6$

$\Rightarrow A = 1,\ B = 4.$

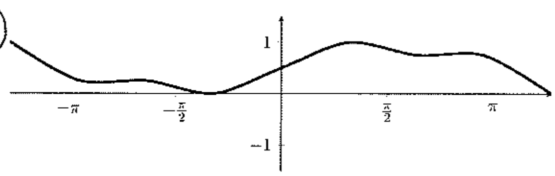
$V(x) = x + 4$

10. (6 points) Each graph below shows a single period of a certain periodic function. Which function will have a Fourier series that contains a nonzero constant term but has all other cosine terms being zero?

(a)

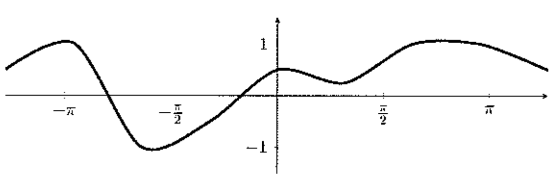


(b)

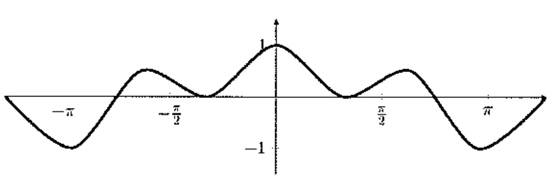


Shifted odd function.

(c)



(d)



11. (12 points) True or false:

(a) Every exact equation can be rewritten into a separable equation.

$\boxed{\text{False}}$ (every separable equation is exact, but not ~~vo~~ vice-versa)

(b) If $\mathcal{L}\{y(t)\} = e^{-12s}\dfrac{s}{s^4+2s+1}$, then $y(10) = 0$.

$\boxed{\text{True}}$

$y = u_{12}(t)\mathcal{L}^{-1}\left\{\frac{s}{s^4+2s+1}\right\}(t-12)$

$\Rightarrow y(10) = 0$ as $u_{12}(10) = 0$.

(c) If $x(t) = \begin{bmatrix} C_1e^{3t} \\ C_2e^{3t} \end{bmatrix}$ is the general solution of the system $x' = Ax$. Then the coefficient matrix must be $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$x(t) = C_1\begin{pmatrix}1\\0\end{pmatrix}e^{3t} + C_2\begin{pmatrix}0\\1\end{pmatrix}e^{3t}$

$\Rightarrow A\begin{pmatrix}1\\0\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix}$, $A\begin{pmatrix}0\\1\end{pmatrix} = 3\begin{pmatrix}0\\1\end{pmatrix}$

$\therefore A = \begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$

$\boxed{\text{True}}$

(d) The Fourier series representing any odd periodic function $f(x)$ necessarily converges to 0 at $x = 0$.

$f(-x) = -f(x) \Rightarrow$ ~~f(0) = -f(0)~~ $\lim_{x\to 0^+} f(-x) = -\lim_{x\to 0^+} f(x)$.

So $\frac{1}{2}\left(\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right) = 0$

$\boxed{\text{True}}$

(e) Using the formula $u(x,t) = X(x)T(t)$, the boundary conditions $u(0,t) = 0$ and $u_x(9,t) = 0$ can be rewritten as $T(0) = 0$ and $T'(9) = 0$.

$\boxed{\text{False}}$

$X(0)T(t) = 0 \Rightarrow X(0) = 0$

$X'(9)T(t) = 0 \quad X'(9) = 0$

(f) $X(x) = 20$ is an eigenfunction of the 2-point boundary value problem

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \qquad X'(1) = 0.$$

$\boxed{\text{True}}$ $\lambda = 0$

12. (16 points) Consider the two-point boundary value problem

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \qquad X(4) = 0.$$

(a) (12 points) Find all **positive** eigenvalues and corresponding eigenfunctions of the boundary value problem.

$\lambda = \sigma^2 > 0.$

3pts $\rightarrow$ $X = C_1 \cos(\sigma x) + C_2 \sin(\sigma x)$ $\Rightarrow$ $C_2 = 0$ &

$X' = -\sigma C_1 \sin(\sigma x) + \sigma C_2 \cos(\sigma x)$ $\quad C_1 \cos(4\sigma) = 0.$

So $4\sigma = \frac{(2n-1)\pi}{2}$ for $n = 1, 2, 3, \ldots$ 6pts for applying BCs & finding λ_n

$\Rightarrow \sigma_n = \frac{(2n-1)\pi}{8}$. So $\lambda_n = \frac{(2n-1)^2\pi^2}{64}$, and

3pts $\rightarrow$ $X_n = \cos\left(\frac{(2n-1)\pi x}{8}\right)$, $n = 1, 2, 3, \ldots$.

(b) (4 points) Is $\lambda = 0$ an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.

No, as $X'' = 0 \Rightarrow X = Ax + B$ w/ $X'(0) = 0$, $X(4) = 0$

$\Rightarrow A = 0$ & $B = 0$.

This leaves only the trivial solution.

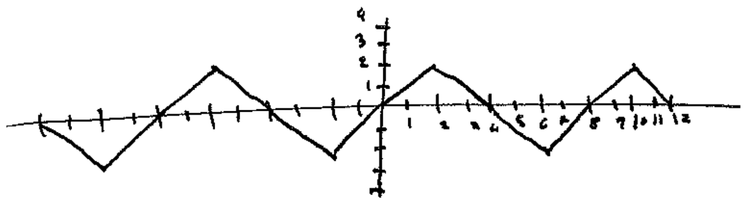
2pts for arriving at the correct conclusion for the evidence they present.

2pts for the correct answer.

13. (17 points) Let

$$f(x) = \begin{cases} x & 0 < x \le 2, \\ 4 - x & 2 < x < 4. \end{cases}$$

(a) (4 points) Consider the **odd** periodic extension, of period $T = 8$, of $f(x)$. Sketch 3 periods, on the interval $-12 < x < 12$, of this function.

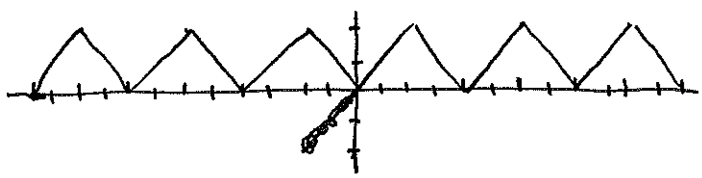


(b) (2 points) To what value does the Fourier series of this odd periodic extension converge at $x = -4$? At $x = 23$?

$f(-4) = f(4) = 0$

$23 = 24 - 1 \Rightarrow f(23) = f(-1) = -f(1) = -1.$

(c) (4 points) Consider the **even** periodic extension, of period $T = 8$, of $f(x)$. Sketch 3 periods, on the interval $-12 < x < 12$, of this function.



(d) (3 points) Find $\frac{a_0}{2}$, the constant term of the Fourier series of the even periodic function described in (c).

$$\frac{a_0}{2} = \frac{1}{4}\int_0^4 f(x)\,dx = \frac{1}{4}\left(\frac{1}{2}(4)(2)\right) = \frac{1}{4}(4) = 1.$$

(e) (4 points) Which of the integrals below can be used to find the Fourier cosine coefficients of the even periodic extension in (c)?

1. $a_n = \frac{1}{2}\left(\int_0^2 x\cos\frac{n\pi x}{4}dx - \int_2^4 (x-4)\cos\frac{n\pi x}{4}dx\right)$
2. $a_n = \frac{1}{4}\left(\int_{-4}^{-2} (x-4)\cos\frac{n\pi x}{4}dx + \int_{-2}^{2} x\cos\frac{n\pi x}{4}dx - \int_2^4 (x-4)\cos\frac{n\pi x}{4}dx\right)$
3. $a_n = \frac{1}{2}\left(\int_0^2 x\cos\frac{n\pi x}{2}dx + \int_2^4 (4-x)\cos\frac{n\pi x}{2}dx\right)$
4. $a_n = \frac{1}{2}\left(\int_0^2 x\sin\frac{n\pi x}{4}dx + \int_2^4 (4-x)\sin\frac{n\pi x}{4}dx\right)$

$$a_n = \frac{2}{L}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right) = \frac{1}{2}\left(\int_0^2 x\cos\left(\frac{n\pi x}{4}\right)dx + \int_2^4 (4-x)\cos\left(\frac{n\pi x}{4}\right)dx\right).$$

14. (16 points) Given the heat conduction equation $\alpha^2 u_{xx} = u_t, \quad 0 < x < 5, \quad t > 0$, and consider the following initial-boundary value problems

I. $u(0,t) = 30, \quad u(5,t) = 30,$
$u(x,0) = 40 - 100\cos(2\pi x).$ $v(x) = 30$

II. $u(0,t) = 30, \quad u(5,t) = 60,$
$u(x,0) = 40 - 100\cos(2\pi x).$ $v(x) = 30 + 6x$

III. $u(0,t) = 60, \quad u(5,t) = 30,$
$u(x,0) = 40 - 100\cos(2\pi x).$ $v(x) = 60 - 6x$

IV. $u_x(0,t) = 0, \quad u_x(5,t) = 0,$
$u(x,0) = 40 - 100\cos(2\pi x).$ $v(x) = 40.$

(a) (3 points) Which problem (**I**, **II**, **III**, or **IV**) models the temperature distribution of a rod with both ends kept at an identical temperature?

I

(b) (3 points) Which problem (**I**, **II**, **III**, or **IV**) models the temperature distribution of a rod where the right end is kept at 60 degrees and the left end kept at 30 degrees temperature?

II

(c) (4 points) Let $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$, and $u_4(x,t)$ be the solutions of their respective initial-boundary value problems. Consider the temperatures at the point $x = 2$ as $t \to \infty$ of each solution. Which initial-boundary value problem has the highest limiting temperature?

(1) **I** 30
(2) **II** 42
(3) **III** 48
(4) **IV** 40

(d) (2 points) What is the actual value of the highest limiting temperature achieved in part (c)?

48

(e) (4 points) Suppose the initial condition is, instead, $u(x,0) = 80 - 40\cos(10\pi x)$. Will your answer to part (c) change? Why or why not?

Yes, IV now takes the values $v(2) = 80$, which is higher than 48.

15. (13 points) Suppose the temperature distribution function $u(x,t)$ of a rod is given by the initial-boundary value problem

$$\begin{aligned} &3u_{xx} = u_t, \qquad 0 < x < 2\pi, \quad t > 0 \\ &u(0,t) = 0, \qquad u(2\pi, t) = 0, \\ &u(x,0) = 25\sin(\tfrac{3}{2}x) + 16\sin(4x) - 9\sin(5x) \end{aligned}$$

(a) (9 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

$$(*)\ u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{3n^2\pi^2 t}{4\pi^2}} \sin\left(\frac{n\pi x}{2\pi}\right) = \sum_{n=1}^{\infty} C_n e^{-\frac{3n^2 t}{4}} \sin\left(\frac{nx}{2}\right).$$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{nx}{2}\right)$$

$(**)$ $C_3 = 25$, $C_8 = 16$, $C_{10} = -9$, all others are zero.

Hence, $(+)$ $u(x,t) = 25e^{-\frac{3(9)t}{4}} \sin\left(\frac{3x}{2}\right) + 16\, e^{-\frac{3(64)t}{4}} \sin(4x) - 9\, e^{-\frac{3(100)t}{4}} \sin(5x)$

4 pts for general solution $(*)$
3 pts for finding coefficients $(**)$
2 pts for plugging in $(+)$

(b) (2 points) Find $\lim_{t\to\infty} u(x,t)$, $0 < x < 2\pi$.

$\lim_{t\to\infty} u(x,t) = 0$.

(c) (2 points) Suppose the initial condition was changed to $u(x,0) = 10 - x + 20\cos(2x)$. What is $\lim_{t\to\infty} u(x,t)$ in this case?

$\lim_{t\to\infty} u(x,t) = 0$, due to boundary conditions.

16. (16 points) Suppose the displacement $u(x,t)$ of a piece of flexible string is given by the initial-boundary value problem

$$\begin{aligned} &64u_{xx} = u_{tt}, \qquad 0 < x < 2, \quad t > 0 \\ &u(0,t) = 0, \quad u(2,t) = 0, \\ &u(x,0) = h(x), \\ &u_t(x,0) = j(x). \end{aligned}$$

(a) (2 points) What is the physical meaning of $h(x)$?

~~Initial~~ Initial displacement of the string.

(b) (2 points) What is the standing waves' speed of propagation along the string?

$\sqrt{64} = 8$

(c) (2 points) TRUE or FALSE: $u(x,t) = 0$ is a possible solution of this problem.

False

(d) (4 points) What can you say about the problem's initial conditions given that the problem has general solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(4n\pi t) \sin\frac{n\pi x}{2}\ ?$$

(1) $h(x) = 0$ and $j(x) \neq 0$.
(2) $h(x) \neq 0$ and $j(x) = 0$.
(3) $h(x) \neq 0$ and $j(x) \neq 0$.

(e) (4 points) Fill in the blanks: The coefficients of the solution in part (d) above can be found using the integral

$$C_n = \frac{2}{\alpha}\int_{\beta}^{\gamma} F(x) \sin\frac{n\pi x}{2} dx.$$

$\alpha =$ $8\pi n$, $\beta =$ 0, $\gamma =$ 2,

$F(x) =$ $j(x)$. (Give $F(x)$ in terms of $h(x)$ and/or $j(x)$.)

(f) (2 points) Does $\lim_{t\to\infty} u(x,t)$ exist? What is it?

No.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	t^n, $\quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}$
4.	t^p, $\quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\dfrac{a}{s^2+a^2}$
6.	$\cos at$	$\dfrac{s}{s^2+a^2}$
7.	$\sinh at$	$\dfrac{a}{s^2-a^2}$
8.	$\cosh at$	$\dfrac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$, $\quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
16.	$(f * g)(t) = \displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$