MATH 251

FINAL EXAMINATION

December 17, 2008

Name:________________

Student Number:________________

Section:________________

This exam has 17 questions for a total of 60 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number. A list of Laplace transforms is attached as the last page of this booklet. It can be removed for easy reference during the examination.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

Do not write in this box.

1-12:	
13:	
14:	
15:	
16:	
17:	
Total:	

1. B
2. A
3. C
4. C
5. A
6. C
7. B
8. D
9. B
10. B
11. D
12. D
13. (a) True
    (b) True
    (c) False
    (d) True
    (e) False
    (f) True

14. (14 points) In each part below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) (4 points) Suppose that the system's general solution is

$$\mathbf{x}(t) = C_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-6t}.$$

Classify the type and stability of the system's critical point at $(0, 0)$.

NODE, ASYMPTOTICALLY STABLE

(b) (4 points) Suppose the only eigenvalue of the coefficient matrix $\mathbf{A}$ is 2, which has corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Write down the system's general solution.

$$y = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(c) (3 points) Classify the type and stability of the critical point at $(0, 0)$ for the system described in (b).

PROPER NODE, UNSTABLE

(d) (3 points) Suppose $\mathbf{A}$ has eigenvalues $9i$ and $-9i$, classify the type and stability of the system's critical point at $(0, 0)$.

CENTER, STABLE

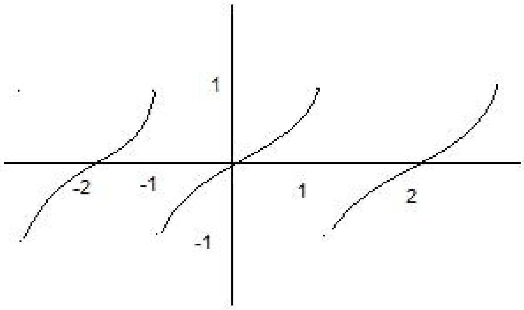
15. (14 points) Find all **positive** eigenvalues and corresponding eigenfunctions of the boundary value problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X'(4) = 0.$$

General Solution: $X = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$ (2pt)
$X' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$ (2pt)
$X(0) = c_1 = 0$ (1pt)
$X'(4) = -c_1\sqrt{\lambda}\sin 4\sqrt{\lambda} + c_2\sqrt{\lambda}\cos 4\sqrt{\lambda} = 0$ (1pt)
$c_1 = 0 \Rightarrow c_2 \neq 0, \lambda \neq 0 \Rightarrow \cos 4\sqrt{\lambda} = 0$ (2pt)
$4\sqrt{\lambda} = n\pi/2$, n odd (2pt)
Eigenvalues: $\lambda = (n\pi/8)^2$, n odd (2pt)
Eigenfunctions: $X = \sin\frac{n\pi x}{8}$, n odd (2pt)

16. (16 points) Let $f(x) = x^3, \qquad 0 < x < 1.$

(a) (4 points) Consider the **odd** periodic extension, of period $T = 2$, of $f(x)$. Sketch 3 periods, on the interval $-3 < x < 3$, of this odd periodic extension.



(b) (2 points) Find a_{10}, the 10th cosine coefficient of the Fourier series of the odd periodic extension in (a).

$a_{10} = 0$

(c) (6 points) Which of the integrals below can be used to find the Fourier sine coefficients of the odd periodic extension in (a)?

$$(i) \qquad b_n = \frac{1}{2}\int_0^1 x^3 \sin\frac{n\pi x}{2}\,dx$$

$$(ii) \qquad b_n = \int_{-1}^1 x^3 \sin\frac{n\pi x}{2}\,dx$$

$$(iii) \qquad b_n = 2\int_0^1 x^3 \sin(n\pi x)\,dx$$

$$(iv) \qquad b_n = \int_{-1}^1 x^3 \cos(n\pi x)\,dx$$

(d) (4 points) To what value does the Fourier series converge at $x = -1$? At $x = \frac{1}{2}$?

0; 1/8

17. (16 points) Suppose the temperature distribution function $u(x,t)$ of a rod that has both ends perfectly insulated is given by the initial-boundary value problem

$$\begin{aligned} &9u_{xx} = u_t, \qquad 0 < x < 4, \qquad t > 0 \\ &u_x(0,t) = 0, \qquad u_x(4,t) = 0, \\ &u(x,0) = 2 - \cos(\pi x) - 7\cos(5\pi x). \end{aligned}$$

(a) (14 points) Find the particular solution of the above initial-boundary value problem.

Assume $u(x,t) = X(x)T(t)$

$9X''T = XT'$

$\frac{X''}{X} = \frac{T'}{9T} = -\lambda$

$X'' + \lambda X = 0$, $T' + 9\lambda T = 0$ (2pt)

$u_x(0,t) = 0, u_x(4,t) = 0 \Rightarrow X'(0)T(t) = 0, X'(4)T(t) = 0 \Rightarrow X'(0) = 0, X'(4) = 0$ (2pt)

$\lambda_n = (n\pi/4)^2$ (2pt)

$X_n = \cos(n\pi x/4)$ (2pt)

$T_n = c_n e^{-9(n\pi/4)^2 t}$ (2pt)

$u(x,t) = c_0/2 + \sum_{n=1}^{\infty} c_n e^{-9(n\pi/4)^2 t}\cos(n\pi x/4)$ (1pt)

$u(x,0) = 2 - \cos(\pi x) - 7\cos(5\pi x) = c_0/2 + \sum_{n=1}^{\infty} c_n \cos(n\pi x/4)$

$c_0 = 4, c_4 = -1, c_{20} = -7$; otherwise $c_n = 0$ (2pt)

Solution: $u(x,t) = 2 - e^{-9\pi^2 t}\cos(\pi x) - 7e^{-9(5\pi)^2 t}\cos(5\pi x)$ (1pt)

(b) (2 points) What is $\lim_{t\to\infty} u(3,t)$?

$\lim = 2$