MATH 251
Fall 2003
Exam II Laplace Transforms Make-Up Exam Solutions

6. (14 pts) Write the following function in terms of unit step functions, and find its Laplace transform.

$$g(t) = \begin{cases} t^2+1 & 0 \le t < 1 \\ e^{-3t}+1 & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases}$$

$$\begin{aligned} g(t) &= (1-u_1(t))(t^2+1) + (u_1(t)-u_2(t))(e^{-3t}+1) + u_2(t) \\ &= t^2+1+u_1(t)(e^{-3t}+1-t^2-1)+u_2(t)(1-e^{-3t}-1) \\ &= t^2+1+u_1(t)(e^{-3t}-t^2)-u_2(t)e^{-3t} \end{aligned}$$

$$\begin{aligned} G(s) = \mathcal{L}\{g(t)\} &= \frac{2}{s^3}+\frac{1}{s}+e^{-s}\mathcal{L}\{e^{-3(t+1)}-(t+1)^2\}-e^{-2s}\mathcal{L}\{e^{-3(t+2)}\} \\ &= \frac{2}{s^3}+\frac{1}{s}+e^{-s}\mathcal{L}\{e^{-3}e^{-3t}-t^2-2t-1\}-e^{-2s}\mathcal{L}\{e^{-6}e^{-3t}\} \\ &= \frac{2}{s^3}+\frac{1}{s}+e^{-s}[\frac{e^{-3}}{s+3}-\frac{2}{s^3}-\frac{2}{s^2}-\frac{1}{s}]-e^{-2s}\frac{e^{-6}}{s+3} \end{aligned}$$

8. (12 pts) Find the inverse Laplace transform of:

$$F(s) = \frac{s^2 - 4}{s^3 + 6s^2 + 9s}$$

First simplify $F(s)$ using partial fractions:

$$F(s) = \frac{s^2 - 4}{s(s^2 + 6s + 9)} = \frac{s^2 - 4}{s(s+3)^2} = \frac{a}{s} + \frac{b}{s+3} + \frac{c}{(s+3)^2} = \frac{a(s+3)^2 + bs(s+3) + cs}{s(s+3)^2}.$$

Therefore,

$$s^2 - 4 = a(s^2 + 6s + 9) + b(s^2 + 3s) + cs = (a+b)s^2 + (6a + 3b + c)s + 9a.$$

Solving the system

$$\begin{aligned} &1 = a + b \\ &0 = 6a + 3b + c \\ &-4 = 9a \end{aligned}$$

$$a = \frac{-4}{9}, b = \frac{13}{9}, c = \frac{-5}{3}.$$

Hence,

$$F(s) = -\frac{4}{9}\frac{1}{s} + \frac{13}{9}\frac{1}{s+3} - \frac{5}{3}\frac{1}{(s+3)^2}.$$

Finally,

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{-4}{9} + \frac{13}{9}e^{-3t} - \frac{5}{3}te^{-3t}.$$

9. (20 pts) Solve the following initial value problem:

$$y'' + 4y' + 8y = e^{2t} - 2\delta(t - 2\pi), \qquad y(0) = 2, \qquad y'(0) = 0$$

Take the Laplace transforms of both sides and simplify:

$$[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + 4[s\mathcal{L}\{y\} - y(0)] + 8\mathcal{L}\{y\} = \frac{1}{s-2} - 2e^{-2\pi s}$$

$$(s^2 + 4s + 8)\mathcal{L}\{y\} - 2s - 8 = \frac{1}{s-2} - 2e^{-2\pi s}$$

$$\mathcal{L}\{y\} = \frac{1}{(s-2)(s^2+4s+8)} + \frac{2s+8}{(s^2+4s+8)} - e^{-2\pi s}\frac{2}{(s^2+4s+8)}$$

$$= \frac{2s^2 + 4s - 15}{(s-2)(s^2+4s+8)} - e^{-2\pi s}\frac{2}{(s^2+4s+8)}$$

We will find the inverse transform of the right-hand side in 2 parts. First note that the $(s^2 + 4s + 8)$ part of the denominators is an irreducible quadratic and proceed accordingly. By completing the squares, it can be rewritten as $s^2 + 4s + 8 = (s + 2)^2 + 2^2$.

Part I.

By partial fractions, $\dfrac{2s^2 + 4s - 15}{(s-2)(s^2+4s+8)} = \dfrac{a}{s-2} + \dfrac{bs+c}{s^2+4s+8}$.

Solve the equation above to get: $a = \dfrac{1}{20}$, $b = \dfrac{39}{20}$, and $c = \dfrac{154}{20}$.

Therefore,

$$\mathcal{L}\{y\} = \frac{1}{20}\frac{1}{s-2} + \frac{1}{20}\frac{39s+154}{s^2+4s+8} = \frac{1}{20}\frac{1}{s-2} + \frac{1}{20}\frac{39(s+2)+76}{(s+2)^2+2^2}$$

$$\mathcal{L}\{y\} = \frac{1}{20}\frac{1}{s-2} + \frac{39}{20}\frac{s+2}{(s+2)^2+2^2} + \frac{38}{20}\frac{2}{(s+2)^2+2^2}.$$

The inverse transform is

$$y_1(t) = \frac{1}{20}e^{2t} + \frac{39}{20}e^{-2t}\cos 2t + \frac{38}{20}e^{-2t}\sin 2t.$$

Part II.

$$-e^{-2\pi s}\frac{2}{(s^2+4s+8)} = -e^{-2\pi s}\frac{2}{(s+2)^2+2^2}$$

Its inverse transform is $y_2(t) = -u_{2\pi}(t)f(t - 2\pi)$.

Where $f(t) = \mathcal{L}^{-1}[\dfrac{2}{(s+2)^2+2^2}] = e^{-2t}\sin 2t$.

Therefore, $y_2(t) = -u_{2\pi}(t)e^{-2(t-2\pi)}\sin 2(t - 2\pi)$.

The very last portion can be further simplified by the identity: $\sin 2(t - 2\pi) = \sin 2t$.

Finally, $y(t) = y_1(t) + y_2(t) = \dfrac{1}{20}e^{2t} + \dfrac{39}{20}e^{-2t}\cos 2t + \dfrac{19}{10}e^{-2t}\sin 2t - u_{2\pi}(t)e^{-2(t-2\pi)}\sin 2t.$