Math 251   March 31, 2005   Second Exam

ANS KEY

1. In Parts **a** and **b** determine the form of a particular solution $y_p = y_p(t)$ having the **least** number of unknown constants. **DO NOT DETERMINE** the unknown constants appearing in your answers in Parts **a** and **b**.

a. 2pt $y'' - 14y' + 49y = 2t^2 e^{7t}$

ANS. $t^2(At^2 + Bt + C)e^{7t}$

7 is a double root of the characteristic polynomial.
Take off 1pt more missing factor of t^2.
Take off 2pts for any other error.

b. 2pt $y'' - 50y' + 49y = 3te^t$

ANS. $t(At + B)e^t$

1 is a simple root of the characteristic polynomial.
Take off 1pt more missing factor of t.
Take off 2pts for any other error.

c. 7pt Without using Laplace transforms, find a particular solution to the following ODE:

$$y'' + 3y = e^t \sin 2t$$

(In this part you need to **determine the unknown constant(s) in the solution.**

ANS. Solve the complexified equation and then take imaginary part

$$y'' + 3y = e^{(1+2i)t}$$

Plug in $y_C = Ae^{(1+2i)t}$ since $1+2i$ is not a root of the characteristic polynomial. (2pts)
$y_C' = A(1+2i)e^{(1+2i)t}$ and $y_C'' = A(-3+4i)e^{(1+2i)t}$.(1pt)
Therefore, $L[y_C] = A(3 + (-3+4i))e^{(1+2i)t}$. (2pt)

We conclude that $A = 1/(4i) = -i/4$ and hence the imaginary part of y_C is $y_p = (-1/4)\cos(2t)$ (2pt)

Alternatively,

plug in $y_p = e^t(A\cos(2t) + B\sin(2t))$. (2pt)
Taking the derivative: $y_p' = e^t((A+2B)\cos(2t) + (B-2A)\sin(2t))$. (1pt)
and again: $y_p'' = e^t((-3A+4B)\cos(2t) + (-3B-4A)\sin(2t))$. (2pt)
Therefore, $L[y_p] = e^t(4B\cos(2t) - 4A\sin(2t))$. (1pt)
We conclude that $4B = 0$ and $-4A = 1$, ie, $y_p = (-1/4)\cos(2t)$. (1pt)

2. Assume that acceleration due to gravity g is equal to 10 meter/sec^2. An object with mass 2 kg stretches a spring 2.5 meters to the equilibrium position. Assume that there is no damping device attached and also assume that at time $t = 0$ the object is released 1 meter below its equilibrium position with a upward velocity of 4 meter/sec.

a. 3pt Write down a differential equation with initial conditions for $y(t)$ for the displacement of the object below its equilibrium position.

ANS. $mg = kL$ Therefore $k = 8$ (1pt)

$2y'' + 8y = 0$ (1pt)

$y(0) = 1$ $y'(0) = -4$ (1pt)

b. 4pt Find a formula for $y(t)$

ANS. $y = c_1 \cos 2t + c_2 \sin 2t$ (2pt)
$y' = -2c_1 \sin 2t + 2c_2 \cos 2t$ (1pt)
$c_1 = 1$ and $c_2 = -2$ (1pt)

c. 2pt Find the maximum value of $y(t)$.

ANS. $R = \sqrt{c_1^2 + c_2^2} = \sqrt{5}$ (2pt)

d. 2pt If a periodic external force equal to $3 \cos \omega t$ Newtons is applied, then for what positive value of ω does resonance occur?

ANS. ω = natural frequency = 2 (2pt)

3. **a. 4pts** For a spring-mass system system with mass equal to 1 kg, spring contant equal to 25 Newton-sec/meter, which damping constant γ causes critical damping?

ANS. Critical damping $= \sqrt{4mk} = \sqrt{(4)(1)(25)} = 10$

b. 3pts If the damping constant γ in the above system is set to 2 Newton-sec/meter, then what can be said about the number of times does the object pass through its equilibrium position?

ANS. Since 2 is less than critical damping, infinitely many times.

c. 4pts If the damping constant γ in the above system is set to 8 Newton-sec/meter, then what is the interval of time between the second time the object returns to its equilibrium position and the third time it returns to its equilibrium position?

ANS. $y'' + 8y' + 25y = 0$ has characteristic polynomial $r^2 + 8r + 25$ which has roots $r = -4 \pm 3i$. (2pt)

The object returns to equilibrium when $c_1 \cos 3t + c_2 \sin 3t$ is zero. (1pt)
This happens at intervals of constant length $\pi/3$. (1pt)

4. **a. 2pts** What is the definition of the Laplace transform $\mathcal{L}\{e^{3t}\}$?

ANS. $\mathcal{L}\{e^{3t}\} = \int_0^\infty e^{-st} e^{3t}\, dt$ (2pt)
No credit for any errors.

b. 4pts Use the answer to Part **a** to calculate $\mathcal{L}\{e^{3t}\}$. (Be sure to explain why this exists only when $s > 3$).

ANS. $\int_0^\infty e^{-st} e^{3t}\, dt = \int_0^\infty e^{-(s-3)t}\, dt$ (1pt)

$\int_0^\infty e^{-(s-3)t}\, dt = \lim_{A\to\infty} \frac{-1}{s-3}\left(e^{-(s-3)A} - 1\right)$ (2pt)

The limit on the right hand side is $\frac{-1}{(s-3)}$ whenever $-(s-3)$ is negative, or equivalently, when $s > 3$. (1pt)

c. 2pts Suppose that the Laplace transform of y is Y. If $y(0) = 2$ and $y'(0) = -3$, then find the Laplace transform of y''.

ANS. $\mathcal{L}\{y'\} = sY - 2$ (1pt)
$\mathcal{L}\{y''\} = s(sY - 2) - (-3) = s^2Y - 2s + 3$ (1pt)

d. 3pts Find the function $f(t)$ whose Laplace transform is equal to $\frac{s}{s^2 + 2s + 5}$

ANS. $\frac{s}{s^2+2s+5} = \frac{s}{(s+1)^2 + 2^2}$ (1pt)
$\frac{s}{s^2+2s+5} = \frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}$ (1pt)
$= \mathcal{L}\{e^{-t}(\cos(2t) - (1/2)\sin(2t))\}$ (1pt)

5. **a. 5pt** Find $f(t)$ so that $\mathcal{L}\{f(t)\} = \dfrac{e^{-3s}}{(s-1)(s+3)}$

ANS.

$$\frac{1}{(s-1)(s+3)} = \frac{1/4}{s-1} - \frac{1/4}{s+3} \text{ (2pt)}$$

$$\frac{1/4}{s-1} - \frac{1/4}{s+3} = \mathcal{L}\{u(t)((1/4)e^{t}) - (1/4)e^{-3t}\} \text{ (1pt)}$$

$$\frac{e^{-3s}}{(s-1)(s+3)} = \mathcal{L}\{u(t-3)((1/4)e^{(t-3)} - (1/4)e^{-3(t-3)})\} \text{ (2pt)}$$

b. 6pt Assume that acceleration due to gravity g is equal to 10 meter/sec^2. An object with mass 2 kg stretches a spring 4 m to equilibrium. At time $t = 0$ it is released 2 meters below its equilibrium position with an upward velocity of 3 meters/sec . At time $t = 6$ it is struck with a hammer and as a result its momentum is **decreased** by 7 kg-meters/sec at that moment in time. At time $t = 8$ a constant external force of 9 Newtons is added and at $t = 10$ it is removed. Write down a differential equation with initial conditions for the displacement $y(t)$ of the object below its equilibrium position. **DO NOT SOLVE THIS EQUATION**

ANS. $2y'' + 5y = -7\delta(t-6) + 9(u(t-8) - u(t-10)) \quad y(0) = 2, \quad y(0) = -3$

(1pt) for the lhs of the ODE.
(2pt) for each of the two terms on the rhs of the ODE.
(1pt) for the initial conditions.

6. **a. 3pt** Consider the function

$$f(t) = \begin{cases} t, & \text{if} \quad t < 2 \\ 2, & \text{if} \quad 2 \le t \end{cases}$$

Sketch a graph of this function and find a formula for $f(t)$ in terms of unit step functions $u(t-c)$, for appropriate values of c. (Note that $u(t-c)$ and $u_c(t)$ denote the same function.)

ANS. $f(t) = t((u(t) - u(t-2)) + 2u(t-2)$

(1pt) for the graph of $f(t)$
(1pt) for each of the two terms in the above expression for $f(t)$.

b. 4pt Determine the Laplace transform of $f(t)$ in Part **a**

ANS. Note that $f(t) = tu(t) - (t-2)u(t-2))$ (1pt)

$\mathcal{L}\{tu(t)\} = 1/s^2$ (1pt)

$\mathcal{L}\{(t-2)u(t-2)\} = e^{-2s}/s^2$ (2pt)

c. 4pt Find the Laplace transform of $u(t-\pi)\sin(t)$.

ANS. Note that $\cos(t-\pi) = \sin(t)$ (2pt)
$\mathcal{L}\{u(t-\pi)\cos(t-\pi)\} = e^{-\pi s}\dfrac{s}{s^2+1}$ (2pt)

7. 11pt Solve the following IVP:

$$y' + 3y = \delta(t-1) + u(t-2) \qquad\qquad y(0) = -4$$

ANS.

Take the Laplace transform of both sides: $sY + 4 + 3Y = e^{-s} + \dfrac{e^{-2s}}{s}$

(1pt) for the lhs
(1pt) for Laplace of delta
(1pt) for Laplace of unit step fcn.

Hence, solving for Y $Y = \dfrac{-4}{s+3} + e^{-s}\dfrac{1}{s+3} + e^{-2s}\dfrac{1}{(s+3)s}$

(1pt) for solving for Y

$\dfrac{1}{(s+3)s} = \dfrac{1/3}{s} - \dfrac{1/3}{s+3}$ (1pt)

Therefore,
$y(t) = -4e^{-3t} + u(t-1)e^{-3(t-1)} + \frac{1}{3}\left(1 - e^{-3(t-2)}\right)u(t-2)$

(1pt) for the first term in $y(t)$
(2pt) for the second term in $y(t)$
(3pt) for the third term in $y(t)$

8. **a.** **7pt** Find the general solution of $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. (Note: the solution should be real.)

ANS. The characteristic polynomial is $(1-r)(1-r)+4$.
The complex eigenvalues are $1 \pm 2i$. (2pt)

Then $A-(1-2i)I = \begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix}$ and hence the corresponding eigenvector is $\begin{pmatrix} 2 \\ 2i \end{pmatrix}$ or more simply $\begin{pmatrix} 1 \\ i \end{pmatrix}$. (2pt)

A complex solution is

$$e^t e^{2it} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t(\cos 2t + i \sin 2t) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

(1pt)

We obtain real solutions by taking the real and imaginary parts of the above:

$$\mathbf{x}_1 = e^t \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} \qquad \mathbf{x}_2 = e^t \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

(2pt)

b. **2pt** Sketch a phase portrait for the system given in Part a.

ANS. The phase portrait is an expanding spiral. (1pt)
To get its orientation check $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, ie, oriented counter-clockwise. (1pt)

c. **1pt** Which of the six names for the critical points fits the critical point of this system.

ANS. Spiral

d. **1pt** What is its stability?

ANS. Unstable

9. In Parts **a** and **b** of this Problem do the following:
i. Sketch a phase portrait for this system.
ii. State the name associated with the critical point at $(0,0)$ and state whether it is stable, asymptotically stable or unstable?

a. 4pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:
$\mathbf{x} = c_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

ANS.

Total of eight trajectories required.
(1pt) for the "basic four" corresponding to the eigenvectors.
(1pt) for the remaining ones
Node (1pt)
Asymptotically stable (1pt)

b. 4pt The homogeneous linear system $\mathbf{x}' = A\mathbf{x}$ whose general solution is:
$\mathbf{x} = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Total of eight trajectories required.
(1pt) for the "basic four" corresponding to the eigenvectors.
(1pt) for the remaining ones
Node (1pt)
Unstable saddle (1pt)

In parts **c.** and **d.** of this Problem do the following:
determine the stability and the name associated with the critical point of the system at the origin.

c. 2pt $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

ANS. Improper node.(1pt)
Unstable (1pt)

d. 2pt $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

ANS. Proper node.(1pt)
Unstable (1pt)