

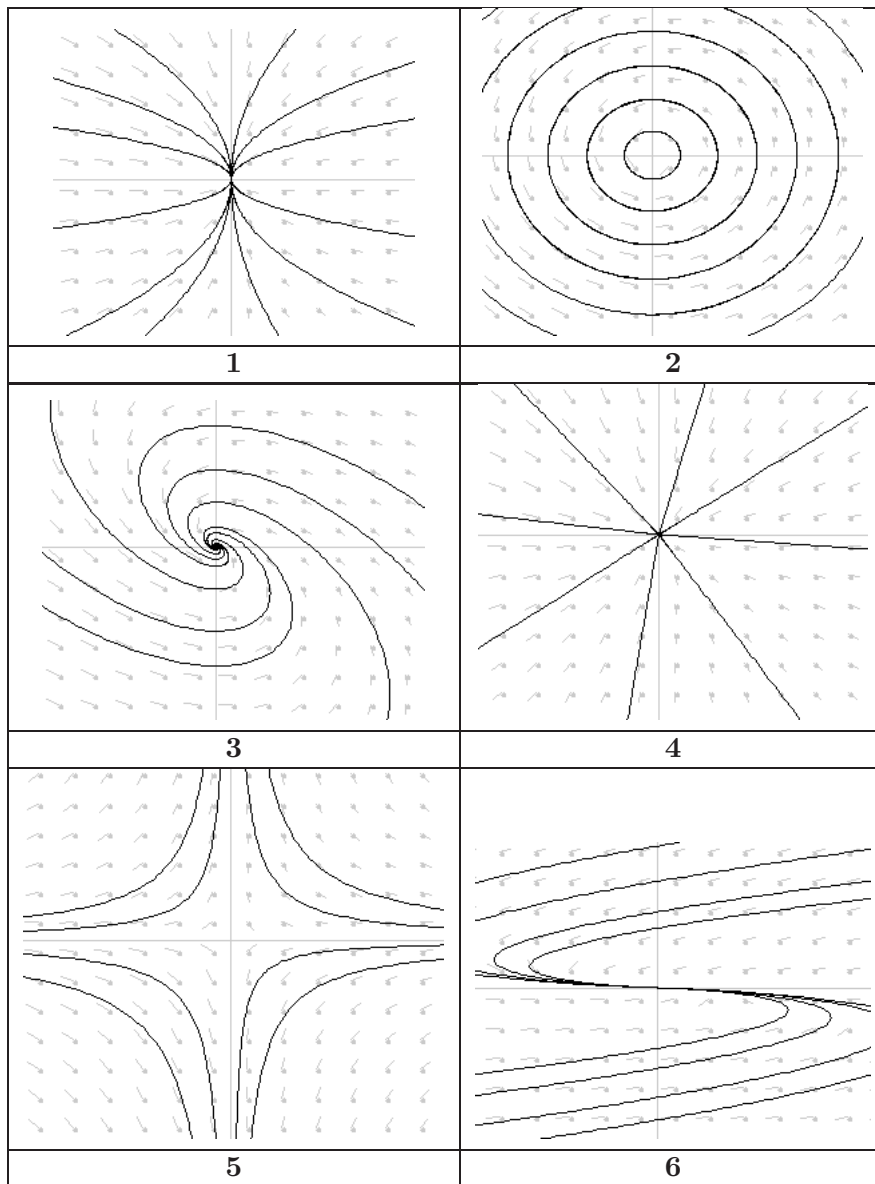
This exam has 13 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM.
PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

Do not write in this box.

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1. (5 points) Match the sketches of phase portraits for 2×2 homogeneous linear systems $\mathbf{x}' = A\mathbf{x}$ with the names of their critical points at the origin.



- | | |
|-----------------|-------|
| (a) saddle | — 5 — |
| (b) node | — 1 — |
| (c) proper node | — 4 — |
| (d) center | — 2 — |
| (e) spiral | — 3 — |

2. (5 points) Match the following formulas for general solutions of 2x2 homogeneous linear systems $\mathbf{x}' = A\mathbf{x}$ with the sketches of the phase portraits given in Problem 1:

- (a) $c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **- 5 -**
- (b) $c_1 e^t \begin{pmatrix} \cos t \\ 2 \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ 2 \cos t \end{pmatrix}$ **- 3 -**
- (c) $c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **- 4 -**
- (d) $c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **- 5 -**
- (e) $c_1 \begin{pmatrix} \cos t \\ 2 \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ 2 \cos t \end{pmatrix}$ **- 2 -**

3. (5 points) Match the three adjectives for the critical point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of a 2×2 homogeneous linear systems $\mathbf{x}' = A\mathbf{x}$ with the five general solutions given in the table below by placing one of the letters **A**, **U**, or **S** in each of the five blanks.

- | | | |
|---|--------------|------------------------------------|
| (a) $c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | - U - | Use |
| (b) $c_1 e^t \begin{pmatrix} \cos t \\ 2 \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ 2 \cos t \end{pmatrix}$ | - U - | A for asymptotically stable |
| (c) $c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | - A - | U for unstable |
| (d) $c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | - U - | S for stable |
| (e) $c_1 \begin{pmatrix} \cos t \\ 2 \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ 2 \cos t \end{pmatrix}$ | - S - | |

4. (5 points) When an object with mass 5 kg is attached to a spring, the object stretches the spring by 2 m. A damper with damping coefficient of 4 N-s/m is attached to the system. Assume there is no external force acting on the system and that acceleration due to gravity is 10 m/s^2 . If the object is released 1 m above its equilibrium position and is given an initial downward velocity of 3 m/s. Assuming that the downward direction is positive, which initial value problem describes the motion of the system?

(a) $5y'' + 4y' + 25y = 0, \quad y(0) = 1, \quad y'(0) = -3$

(b) $5y'' + 4y' + 25y = 0, \quad y(0) = -1, \quad y'(0) = 3$

(c) $5y'' + 4y' + 50y = 0, \quad y(0) = -1, \quad y'(0) = 3$

(d) $5y'' + 4y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 3$

5. (5 points) Initially an object with mass 2 kg is located on a frictionless surface and is moving eastward with no forces acting on it. At time $t = 3$ seconds a constant external eastward force of .17 Newtons is added. At $t = 4$ seconds the object is struck with a hammer in such a fashion that its momentum is reduced by .34 kg-m/s at that time. Which of the following $F(t)$ represents the combination of these two forces.

(a) $F(t) = .17u(t - 3) + .34\delta(t - 4)$

(b) $F(t) = .17\delta(t - 3) + .34u(t - 4)$

(c) $F(t) = .17u(t - 3) - .34\delta(t - 4)$

(d) $F(t) = -.17\delta(t - 3) + .34u(t - 4)$

6. (5 points) What is the general real-valued solution of the following ODE:

$$y''' + y' = 0$$

(a) $y(t) = c_1 e^t + c_2 e^{-t} + c_3$

(b) $y(t) = c_1 \cos t + c_2 \sin t + c_3 e^t$

(c) $y(t) = c_1 + c_2 \cos t + c_3 \sin t$

(d) $y(t) = c_1 t^2 + c_2 t + c_3$

7. (5 points) A spring-mass system is described by the differential equation $y'' + 2y' + 10y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = -1$. Which statement will best describe the behavior of the solution $y(t)$ after a long time?

- (a) $y(t)$ will oscillate with increasing amplitude.
- (b) $\lim_{t \rightarrow \infty} y(t) = 0$
- (c) $y(t)$ will approach a periodic function with period π .
- (d) $y(t)$ will approach a periodic function with period $\frac{2\pi}{3}$.

8. (5 points) $\int_0^{\infty} e^{-(s-2)t} dt$ is the Laplace transform of which of the following functions?

- (a) t^2
- (b) e^{2t}
- (c) $u(t - 2)$
- (d) $\delta(t - 2)$

9. (5 points) If $f(t) = u(t - 2) - 2u(t - 3) + 3u(t - 4) - 4u(t - 5)$, what is $f(\pi)$?

- (a) -2
- (b) -1
- (c) 0
- (d) 1

10. A spring-mass system is modeled by the initial value problem

$$2y'' + \gamma y' + 18y = F(t), \quad y(0) = 3, \quad y'(0) = -12.$$

(a) (6 points) If $\gamma = 0$ and $F(t) = 0$, what is the amplitude of displacement?

If $\gamma = 0$ the roots of the characteristic poly are $\pm 3i$.

1pt

In this case the general solution is $y = c_1 \cos 3t + c_2 \sin 3t$

1pt

Also need $y' = 3(-c_1 \sin 3t + c_2 \cos 3t)$

1pt

Plugging in $y(0) = 3$ $y'(0) = -12$ gives $c_1 = 3$, $c_2 = -4$. 1pt

Therefore, the amplitude $R = \sqrt{3^2 + (-4)^2} = 5$ 2pt

(b) (3 points) If $\gamma = 0$ and $F(t) = 2 \cos(\omega t)$, for which value(s) of ω will the system undergo resonance?

If $\gamma = 0$ the system will undergo resonance when the frequency of the forcing function is equal to the natural frequency of the spring mass system. ie, $\omega = 3$

3points. Give credit based on student's determination of natural frequency in previous part regardless of whether that is correct or not

(c) (3 points) If $F(t) = 0$, for which value(s) of γ will the system **not** oscillate? System will oscillate

not oscillate if γ is greater than or equal to critical damping $= \sqrt{4mk} = \sqrt{(2)(4)(18)} = \sqrt{2^4 3^2} = 12$

3points. Take off 1 point for omitting the value 12

11. Compute the following:

(a) (7 points)

$$\mathcal{L}\{e^{3t}u(t-2)\}.$$

One way is to use the first shift formula and the formula

$$\mathcal{L}\{u(t-2)\} = e^{-2s}\frac{1}{s}. \text{ 3pts for getting the correct starting point.}$$

$$\mathcal{L}\{e^{3t}u(t-2)\} = e^{-2(s-3)}\frac{1}{s-3}$$

4pts for applying the shift formula correctly.

Another way of doing this is to observe that

$$e^{3t}u(t-2) = e^6 e^{3(t-2)}u(t-3) \text{ 3pts for this starting point and then apply the second shift formula:}$$

$$\mathcal{L}\{e^{3t}u(t-2)\} = e^6 \mathcal{L}\{e^{3(t-2)}u(t-2)\} = e^6 \frac{e^{-2s}}{s-3}$$

which agrees with the previous answer. 4pts for applying the shift formula correctly.

(b) (7 points)

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+8s+25}\right\}.$$

First observe that

$$\frac{s}{s^2+8s+25} = \frac{s+4}{(s+4)^2+3^2} - 4\frac{1}{(s+4)^2+3^2}$$

2 pts for completing the square and

2 pts for realizing that $s+4$ is required in the numerator in order to apply the first shift formula.

The following two formulas are combined with the first shift formula:

$$\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+3^2} \quad \mathcal{L}\{\sin 3t\} = \frac{3}{s^2+3^2}$$

1 pt for identifying that Laplace transform of sine and cosine are needed here

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+8s+25}\right\} e^{-4t} \left(\cos 3t - \frac{4}{3}\sin 3t\right)$$

2 pts for applying first shift formula correctly.

12. (16 points) Use Laplace transforms to solve the following initial value problem:

$$y' + 3y = tu(t - 2) + \delta(t - 3), \quad y(0) = 1.$$

$$\mathcal{L}\{\delta(t - 2)\} = e^{-2s}$$

2points

$$\mathcal{L}\{u(t - 2)\} = e^{-2s} \frac{1}{s}$$

2points

$$\mathcal{L}\{tu(t - 2)\} = \mathcal{L}\{(t - 2)u(t - 2)\} + \mathcal{L}\{2u(t - 2)\} = e^{-2s} \left(\frac{1}{s^2} + \frac{2}{s} \right)$$

2points

Alternatively

$$\mathcal{L}\{tu(t - 2)\} = -\frac{d}{ds} \frac{e^{-2s}}{s} = e^{-2s} \left(\frac{1}{s^2} + \frac{2}{s} \right)$$

$$\mathcal{L}\{y' + 3y\} = sY - 1 + 3Y$$

2points

Therefore

$$sY - 1 + 3Y = e^{-2s} \left(\frac{1}{s^2} + \frac{2}{s} \right) \quad Y = \frac{1}{s + 3} + e^{-2s} \left(\frac{1}{s^2(s + 3)} + \frac{2}{s(s + 3)} \right)$$

2points

To find the inverse Laplace transform of the above we need to find the following partial fraction expansion:

$$\frac{1}{s^2(s + 3)} + \frac{2}{s(s + 3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 3}$$

Clearing the denominator gives $1 + 2s = As(s + 3) + B(s + 3) + Cs^2$. Setting $s = 0$ gives $B = 1/3$; setting $s = -3$ gives $C = -5/9$ and equating s^2 terms on both sides of the identity gives: $A + C = 0$. So $A = 5/9$.

2points

Therefore,

$$\mathcal{L} \left\{ u(t) \left(\frac{5}{9} - \frac{t}{3} - \frac{5}{9}e^{-3t} \right) \right\} = \frac{1}{s^2(s + 3)} + \frac{2}{s(s + 3)}$$

and hence

$$\mathcal{L} \{ u(t - 2) (5/9 + t/3 + e^{-3t}) \} = e^{-2s} \left(\frac{1}{s^2(s + 3)} + \frac{2}{s(s + 3)} \right)$$

2points

$$\mathcal{L}^{-1}\{Y\} = e^{-3t} + u(t - 2) \left(\frac{5}{9} + \frac{t}{3} - \frac{5}{9}e^{-3t} \right)$$

2points

13. Consider the 2×2 linear homogeneous system:

$$\mathbf{x}' = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

(a) (2 points) Find the vector \mathbf{x}' at the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

2points

(b) (11 points) Find the solution $\mathbf{x}(t)$ that satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

The characteristic polynomial is: $r^2 + 4$ and hence the eigenvalues are $\pm 2i$. 2points

Determine the eigenvector corresponding to the eigenvalue $2i$

$$A + 2iI = \begin{pmatrix} 2i & 4 \\ -1 & 2i \end{pmatrix}$$

Therefore it is: $\begin{pmatrix} 2i \\ 1 \end{pmatrix}$ 2points

A complex solution is

$$e^{2it} \begin{pmatrix} 2i \\ 1 \end{pmatrix} = (\cos 2t + i \sin 2t) \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} 2i \cos 2t - 2 \sin 2t \\ \cos 2t + i \sin 2t \end{pmatrix}$$

2points

Taking the real and imaginary parts gives the general real-valued solution:

$$c_1 \begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix}$$

2points

We plug in the initial conditions and solve find c_1 and c_2 that satisfy the initial conditions:

$$c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Therefore $c_2 = -1/2$ and $c_1 = 0$ 2points

(Obviously, these constants depend on the choice of the eigenvector.)

The final answer is:

$$\mathbf{x} = \begin{pmatrix} -\cos 2t \\ -\frac{1}{2} \sin 2t \end{pmatrix}$$

1 point

Alternate solution for last 5/11 of the problem.

Plugging in the initial condition into the complex general solution

$$\mathbf{x} = d_1 e^{2it} \begin{pmatrix} 2i \\ 1 \end{pmatrix} + d_2 e^{-2it} \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

3points

$$d_1 \begin{pmatrix} 2i \\ 1 \end{pmatrix} + d_2 \begin{pmatrix} -2i \\ 1 \end{pmatrix} = \begin{pmatrix} 2id_1 - 2id_2 \\ d_1 + d_2 \end{pmatrix} = \begin{pmatrix} -\cos 2t \\ -\frac{1}{2} \sin 2t \end{pmatrix}$$

Therefore, $d_1 = 1/4i$ $d_2 = -1/4i$ 2points