MATH 251 Fall 2003 Exam 2

November 10, 2003

Solution Key

1. (5 points) A spring-mass system, subject to an external force of $10\cos(2t)$ Newtons, is equipped with a spring with a Hooke's constant 12 Newtons per meter. For what mass will the resonance occur?

   (a) 2 kg

   (b) 3 kg `Correct`

   (c) 6 kg

   (d) 10 kg

   `Solution:` The frequency of free oscillation ω is subject to the following relation:

   $$\omega^2 = \frac{\text{spring constant}}{\text{mass}}$$

   For resonance the frequency of the external force should be equal to the frequency of free oscillation.

2. (5 points) Of what form will the particular solution to the following differential equation be? Do not solve the equation.

   $$y'' - 4y' + 4y = e^{2t} + t^2e^{3t} - \sin(2\pi t)$$

   (a) $Ae^{2t} + Bt^2e^{3t} + Cte^{3t} + De^{3t} + E\sin(2\pi t) + F\cos(2\pi t)$

   (b) $Ate^{2t} + Bt^2e^{3t} + Cte^{3t} + De^{3t} + E\sin(2\pi t) + F\cos(2\pi t)$

   (c) $At^2e^{2t} + Bt^2e^{3t} + Cte^{3t} + De^{3t} + E\sin(2\pi t) + F\cos(2\pi t)$ `Correct`

   (d) $Ae^{2t} + Bt^2e^{3t} + E\sin(2\pi t) + F\cos(2\pi t)$

   `Solution:` The characteristic equation, associated with the homogeneous part of the equation, has root 2 of multiplicity 2. Therefore the solution-to-be has the e^{2t} term multiplied by t^2. For the t^2e^{3t} part of the right–hand side we need to throw a full polynomial of degree 2 multiplied by the exponent function, not just Bt^2.

3. (5 points) The second order linear equation $y'' - 4y' + 4y = 0$ is equivalent to

   (a) $\begin{cases} x_1' = x_2 \\ x_2' = -4x_1 + 4x_2 \end{cases}$ `Correct`

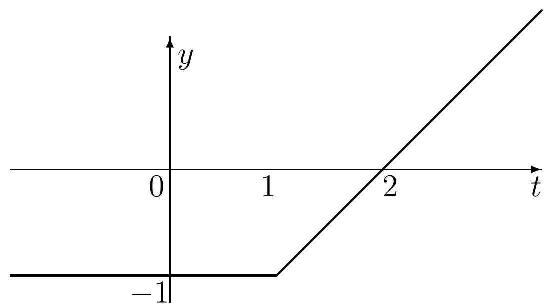
   (b) $\begin{cases} x_1' = x_1 \\ x_2' = -4x_1 + 4x_2 \end{cases}$

   (c) $\begin{cases} x_1' = x_2 \\ x_2' = 4x_1 - 4x_2 \end{cases}$

(d) $\begin{cases} x_1' = x_1 \\ x_2' = 4x_1 - 4x_2 \end{cases}$

`Solution:` We think of x_1 as y and x_2 as y'.

4. (5 points) Which of the following functions corresponds to this graph:



(a) $tu_1(t) - 1$

(b) $tu_1(t-1) - 1$

(c) $(t-1)u_1(t) - 1$ `Correct`

(d) $(t-1)u_1(t-1) - 1$

`Solution:` We are 'turning' at $t = 1$, that's why we must have $u_1(t)$ term. Before the turnpoint, we are at the level -1, that's why we have to add constant -1. After the turnpoint, our function looks like $t - 2$, therefore we write $t - 1$ in front of the Heaviside function (remember, we already have -1 before the turn).

5. (14 points) Solve the following initial value problem:

$$y'' - 5y' - 14y = -14t^2 - 10t - 26, \qquad y(0) = 0, \qquad y'(0) = 13$$

`Solution:` The associated homogeneous differential equation

$$y'' - 5y' - 14y = 0$$

has characteristic polynomial 2 points

$$\lambda^2 - 5\lambda - 14 = (\lambda - 7)(\lambda + 2)$$

with roots 7 and -2, therefore its general solution is given by

$$y_{\text{homogeneous}} = C_1 e^{7t} + C_2 e^{-2t}$$

with C_1 and C_2 arbitrary constants. 2 points

According to the general yoga of the linear differential equations with constant coefficients, we are looking for a particular solution of the nonhomogeneous differential equation

$$y'' - 5y' - 14y = -14t^2 - 10t - 26$$

in the form 4 points

$$y_{\text{particular}} = At^2 + Bt + C$$

with unknown coefficients A, B, and C. Plugging this expression into our equation, we obtain:

$$(2A) - 5(2At + B) - 14(At^2 + Bt + C) = (\text{regroupping}) =$$

$$= (-14A)t^2 + (-10A - 14B)t + (2A - 5B - 14C) = -14t^2 - 10t - 26$$

To make this happen, we need

$$\begin{cases} -14A & = -14 \\ -10A - 14B & = -10 \\ 2A - 5B - 14C & = -26 \end{cases}$$

The first equation gives $A = 1$, then this A and the second equation give $B = 0$, plugging these A and B into the last equation, we have $C = 2$. This means 2 points

$$y_{\text{particular}} = t^2 + 2$$

All solutions of the original equation are then given by 2 points

$$y(t) = y_{\text{homogeneous}} + y_{\text{particular}} = C_1 e^{7t} + C_2 e^{-2t} + t^2 + 2$$

We need to choose constants C_1 and C_2 to satisfy the initial conditions:

$$y(0) = C_1 + C_2 + 2 = 0$$

$$y'(0) = 7C_1 - 2C_2 = 13$$

Solving these algebraic equations for C_1 and C_2, we obtain: 2 points

$$C_1 = 1 \qquad C_2 = -3$$

Finally, the solution to the initial value problem looks like

$$y(t) = e^{7t} - 3e^{-2t} + t^2 + 2$$

`Alternative solution:` We can solve this problem via Laplace transform as well. First, take the transform of both sides of the equation: 2 points

$$\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} - 14\mathcal{L}\{y\} = -14\mathcal{L}\{t^2\} - 10\mathcal{L}\{t\} - 26\mathcal{L}\{1\}$$

Engaging the transform formulæ for derivatives, we obtain: 3 points

$$\text{LHS} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 5s\mathcal{L}\{y\} + 5y(0) - 14\mathcal{L}\{y\}$$

Engaging the transform of the monomial functions, we have: 2 points

$$\text{RHS} = -14\frac{2}{s^3} - 10\frac{1}{s^2} - 26\frac{1}{s}$$

Plugging in the initial conditions, we have: 1 point

$$(s^2 - 5s - 14)\mathcal{L}\{y\} = 13 - 14\frac{2}{s^3} - 10\frac{1}{s^2} - 26\frac{1}{s}$$

Or, equivalently,

$$\mathcal{L}\{y\} = \frac{13s^3 - 26s^2 - 10s - 28}{s^3(s^2 - 5s - 14)}$$

Partial fraction expansion leads to 3 points

$$\mathcal{L}\{y\} = \frac{2}{s^3} + \frac{0}{s^2} + \frac{2}{s} + \frac{1}{s-7} + \frac{-3}{s+2}$$

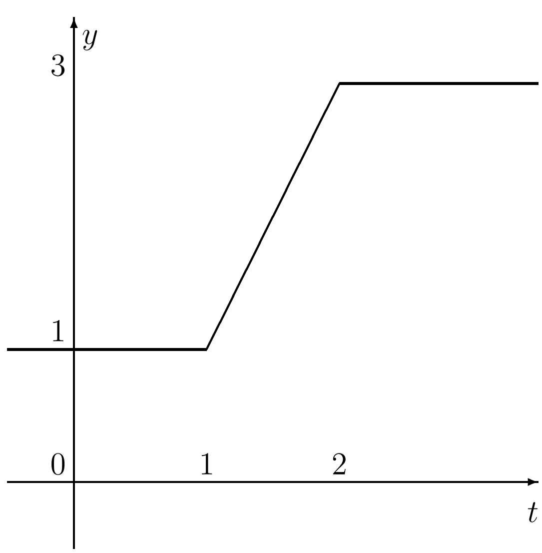
Finally, we need to look at the table of transforms to write down the inverse transform of the sum above: 3 points

$$y(t) = t^2 + 2 + e^{7t} - 3e^{-2t}$$

6. (14 points) Rewrite the following function in terms of step functions and find its Laplace transform.

$$f(t) = \begin{cases} 1, & t < 1 \\ 2t - 1, & 1 \leqslant t < 2 \\ 3, & t \geqslant 2 \end{cases}$$

`Solution:` It makes sense to sketch a graph of this function to see that it is indeed a ramp function:



Now, in terms of step functions, we write 6 points

$$y(t) = 1 + \frac{3-1}{2-1}((t-1)u_1(t) - (t-2)u_2(t)) = 1 + 2(t-1)u_1(t) - 2(t-2)u_2(t)$$

Since the laplace transform of the linear combination of functions is the linear combination of the transforms, we have: 2 points

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{1\} + 2\mathcal{L}\{((t-1)u_1(t))\} - 2\mathcal{L}\{((t-2)u_2(t))\}$$

Using formulæ for the Laplace transform of $(t-a)u_a(t)$ and constant 1, we obtain: 6 points

$$\mathcal{L}\{y(t)\} = \frac{1}{s} + 2e^{-s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} = \frac{s + 2e^{-s} - 2e^{-2s}}{s^2}$$

7. (10 points) Assume that $g = 10\frac{\text{m}}{\text{sec}^2}$ for this problem.
   (a) Mass of 2 kg is weighting down a spring by 80 cm. Calculate the spring constant.
   `Solution:` We are working in the metric system, therefore 80 cm is 0.8 m. 2 points
   Mass of 2 kg implements weight of $2 \cdot 10$ Newtons, and since this force stretches down the spring by 0.8 m, the Hooke's constant is 2 points

$$k = \frac{2 \cdot 10}{0.8} = 25$$

   (b) If the above mentioned spring–mass system is placed into the liquid with damping constant 14 Newton-seconds per meter, and no external force is in effect, what would be the quasi–period of the vibration?
   `Solution:` We know that for an underdamped motion the quasi-frequence is given by the formula 2 points

$$\omega = \frac{\sqrt{4 \cdot \text{spring constant} \cdot \text{mass} - \text{damping constant}^2}}{2 \cdot \text{mass}}$$

   With numbers we have it looks like 2 points

$$\omega = \frac{\sqrt{4 \cdot 25 \cdot 2 - 14^2}}{2 \cdot 2} = \frac{1}{2}$$

   Therefore the quasi–period should be 2 points

$$T = \frac{2\pi}{\omega} = 4\pi$$

8. (12 points) Calculate the inverse Laplace transform of

$$\frac{s^2+9}{9s - s^3}$$

`Solution:` First we split our fraction into simple partial fractions: 6 points

$$\frac{s^2+9}{9s-s^3} = \frac{s^2+9}{-s(s-3)(3+s)} = \frac{1}{s} + \frac{-1}{s-3} + \frac{-1}{s+3}$$

Then, looking at the table of Laplace transforms, we find that 2 points

$$\mathcal{L}\{1\} = \frac{1}{s}$$

and 2 points

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3} \qquad \mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$$

Therefore the inverse transform of our original fraction is 2 points

$$1 - e^{3t} - e^{-3t}$$

9. (20 points) Solve the following initial value problem:

$$y'' + 2y' + 10y = 10 + \delta(t), \qquad y(0) = 1, \qquad y'(0) = 0$$

`Solution:` We calculate the Laplace transform of both sides of our equation: 2 points

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 10\mathcal{L}\{1\} + \mathcal{L}\{\delta(t)\}$$

Engaging the transform formulæ for derivatives, we obtain: 3 points

$$\text{LHS} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} - 2y(0) + 10\mathcal{L}\{y\}$$

Engaging the transform of the Dirac's δ–function, we have: 3 points

$$\text{RHS} = \frac{10}{s} + 1$$

Plugging in the initial conditions, we have: 2 points

$$(s^2+2s+10)\mathcal{L}\{y\} = s + \frac{10}{s} + 3$$

Or, equivalently,

$$\mathcal{L}\{y\} = \frac{s+3}{s^2+2s+10} + \frac{10}{s(s^2+2s+10)}$$

Partial fraction expansion of the last term leads to 4 points

$$\mathcal{L}\{y\} = \frac{s+3}{s^2+2s+10} + \frac{1}{s} + \frac{-s-2}{s^2+2s+10} = \frac{1}{s} + \frac{1}{s^2+2s+10}$$

Now, since $s^2+2s+10 = (s+1)^2+3^2$, we need to rewrite the last fractions in terms of $s+1$ and 3: 2 points

$$\mathcal{L}\{y\} = \frac{1}{s} + \frac{1}{3}\cdot\frac{3}{(s+1)^2+3^2}$$

Finally, we need to look at the table of transforms to write down the inverse transform of the sum above: 4 points

$$y(t) = 1 + \frac{1}{3}e^{-t}\sin(3t)$$

10. (10 points) Solve the initial value problem

$$X' = \begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix} X, \qquad X(0) = \begin{bmatrix} -8 \\ 13 \end{bmatrix}$$

`Solution:` First, we calculate eigenvalues of the matrix $\begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$: 2 points

$$\det \begin{bmatrix} 3-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda) - 3 \cdot 4 = \lambda^2 - 5\lambda - 6 = (\lambda+1)(\lambda-6)$$

So the eigenvalues are -1 and 6. For each eigenvalue we are looking for a eigenvector: 4 points

$$\lambda = -1: \qquad \begin{bmatrix} 3+1 & 3 \\ 4 & 2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\lambda = 6: \qquad \begin{bmatrix} 3-6 & 3 \\ 4 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now the general solution can be written as 2 points

$$X(t) = C_1 \begin{bmatrix} 3 \\ -4 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

In order to determine constants C_1 and C_2, we apply the initial condition:

$$X(0) = \begin{bmatrix} -8 \\ 13 \end{bmatrix} = C_1 \begin{bmatrix} 3 \\ -4 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus

$$\begin{cases} 3C_1 + C_2 & = -8 \\ -4C_1 + C_2 & = 13 \end{cases}$$

$$\begin{cases} C_1 & = -3 \\ C_2 & = 1 \end{cases}$$

Therefore the solution to the initial value problem is 2 points

$$X(t) = -3 \begin{bmatrix} 3 \\ -4 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

`Alternative solution:` We can rewrite our matrix equation as a system of two equations with two unknown functions: 1 point

$$\begin{cases} x_1' & = 3x_1 + 3x_2 \\ x_2' & = 4x_1 + 2x_2 \end{cases} \qquad \begin{cases} x_1(0) & = -8 \\ x_2(0) & = 13 \end{cases}$$

Using the first equation, we can express x_2 in terms of x_1 and its derivative: 1 point

$$x_2 = \frac{1}{3}x_1' - x_1$$

By differentiating this expression, we obtain:

$$x_2' = \frac{1}{3}x_1'' - x_1'$$

On the other hand, the second equation in our system says that

$$x_2' = 4x_1 + 2x_2$$

Plugging in expressions for x_2 and x_2', this equation becomes

$$\frac{1}{3}x_1'' - x_1' = 4x_1 + 2(\frac{1}{3}x_1' - x_1)$$

We write this equation in more conventional way and add the initial condition we have for x_1: 2 points

$$x_1'' - 5x_1' - 6x_1 = 0 \qquad y_1(0) = -8$$

Then we set up the characteristic equation

$$\lambda^2 - 5\lambda - 6 = (\lambda - (-1))(\lambda - 6) = 0$$

It has roots -1 and 6, therefore the general solution to our equation is 2 points

$$x_1 = C_1 e^{-t} + C_2 e^{6t}$$

If we want to satisfy the initial condition, we need to arrange constants C_1 and C_2 such that

$$-8 = C_1 + C_2$$

Now, if we take derivative of our general solution, we obtain

$$x_1' = -C_1 e^{-t} + 6C_2 e^{6t}$$

We know that 2 points

$$x_2 = \frac{1}{3}x_1' - x_1 = \frac{1}{3}(-C_1 e^{-t} + 6C_2 e^{6t}) - (C_1 e^{-t} + C_2 e^{6t}) = -\frac{4}{3}C_1 e^{-t} + C_2 e^{6t}$$

But our second initial condition says that $x_2(0) = 13$, that is

$$13 = -\frac{4}{3}C_1 + C_2$$

Solving the algebraic system

$$\begin{cases} C_1 + C_2 & = -8 \\ -\frac{4}{3}C_1 + C_2 & = 13 \end{cases}$$

we are getting

$$\begin{cases} C_1 & = -9 \\ C_2 & = 1 \end{cases}$$

And this means that the solutions to the original equations are 2 points

$$\begin{cases} x_1 & = -9e^{-t} + e^{6t} \\ x_2 & = 12e^{-t} + e^{6t} \end{cases}$$