This exam has 13 questions for a total of 100 points. Show all your work! In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number.

You may not use a calculator on this exam. Please turn off and put away your cell phone.

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1. (5 points) Consider the differential equation

\[ y' = -2 \sin(2y) \]

Which of following statements is true?
(I) The equation is separable, but not linear.
(II) The general solution is \( y = \cos(2y) + C \).

\[ \frac{\mathrm{d}y}{\sin(y)} = \int -2 \, \mathrm{d}x \quad \text{not linear.} \]

(a) Only (I) is true.
(b) Only (II) is true.
(c) Both are true.
(d) Neither is true.

*Note that if \( y = \frac{\pi n}{2} \) & \( C = (-1)^n + \frac{\pi n}{2} \), this will give a solution, but never the general solution.

2. (5 points) Consider the initial value problem

\[ t \frac{\mathrm{d}y}{\mathrm{d}t} + (t^2 - 1)y = \ln|t - 5|, \quad y(2) = 6. \]

According to the Existence and Uniqueness Theorem, what is the largest interval in which a unique solution is guaranteed to exist?

(a) \((0,5)\)
(b) \((-\infty,5)\)
(c) \((5,\infty)\)
(d) \((1,\infty)\)
3. (5 points) What is a suitable integrating factor that can be used to solve the equation

\[ ty' + (2 - 4t)y = e^{-t} \cos(2t). \]

DO NOT solve this differential equation.

(a) \( \mu(t) = e^{2t-2t} \)
(b) \( \mu(t) = e^{2t} \)
(c) \( \mu(t) = t^{-4}e^{2t} \)
(d) \( \mu(t) = t^2e^{-4t} \)

\[
\mu(t) = \frac{1}{\int p(t) \, dt} = \frac{1}{\int 2-4t \, dt} = \frac{1}{t} \]

\[
\int p(t) \, dt = 2\ln t - 4t + C
\]

\[
\therefore \int p(t) \, dt = 2\ln t - 4t
\]

\[
\therefore \mu(t) = t^2 \cdot e^{-4t}
\]

4. (5 points) Which equation below has the property that some, but not all, of its nonzero solutions converge to zero as \( t \to \infty)?

(a) \( y'' + 2y' + y = 0 \)
(b) \( y'' + 4y' - 5y = 0 \)
(c) \( y'' - 5y' + 6y = 0 \)
(d) \( y'' + 4y = 0 \)
5. (5 points) Which pair of functions below can be a fundamental set of solutions for a second order homogeneous linear equation?

(a) \( y_1 = t^2 - 2t, \quad y_2 = 8t - 4t^2 \)
\[ y_2 = -4y_1 \Rightarrow W(y_1, y_2) = 0 \times \]

(b) \( y_1 = 0, \quad y_2 = \sin 2t - \cos 2t \)
\[ y_1 = 0 \Rightarrow W(y_1, y_2) = 0 \times \]

(c) \( y_1 = 3, \quad y_2 = 3t \)
\[ W = \begin{vmatrix} 3 & 3 \end{vmatrix} - 0 = 9 \neq 0 \checkmark \]
\[ y_1 = e^{10y_2} \Rightarrow W = 0 \times \]

(d) \( y_1 = e^{4t+1}, \quad y_2 = e^{4t-9} \)

6. (5 points) Find the general solution of the fifth order linear equation
\[ y^{(5)} - 6y^{(3)} + 9y' = 0. \]

(a) \( C_1 + (C_2 + C_3t)\sin(\sqrt{3}t) + (C_4 + C_5t)\cos(\sqrt{3}t) \)

(b) \( C_1 + C_2e^{-\sqrt{3}t} + C_3te^{-\sqrt{3}t} + C_4t^3e^{-\sqrt{3}t} + C_5te^{\sqrt{3}t} \)

(c) \( C_1 + C_2e^{-\sqrt{3}t} + C_3te^{-\sqrt{3}t} + C_4e^{\sqrt{3}t} + C_5te^{\sqrt{3}t} \)

(d) \( C_1 + C_2e^{-\sqrt{3}t} + C_3te^{\sqrt{3}t} + C_4 \sin(\sqrt{3}t) + C_5 \cos(\sqrt{3}t) \)

\[ r^5 - 6r^3 + 9r = 0 \]
\[ r(r^4 - 6r + 9) = 0 \]
\[ r(r^2 - 3)^2 = 0 \]
\[ \Rightarrow r = 0, \quad r = \pm \sqrt{3} (\text{twice}) \]
\[ y = C_1 + C_2 e^{\sqrt{3}t} + C_3 t e^{\sqrt{3}t} + C_4 e^{-\sqrt{3}t} + C_5 t e^{-\sqrt{3}t}. \]
7. (8 points) True or false:

(a) \((t^2 + 1)y' = e^{-3t} \arctan(t) y\) is a first order equation that is both linear and separable.
\[
\int \frac{dy}{y} = \int \frac{e^{-3t}}{t^2 + 1} \, dt
\]
True

(b) Every separable equation \(M(t) + N(y) \frac{dy}{dt} = 0\) is also an exact equation.
\[
\text{True as } M_y = 0 = N_t.
\]

(c) Every autonomous equation \(y' = f(y)\) is also a linear equation.
False, \(f(y)\) could be nonlinear.

(d) Given that \(y_1\) and \(y_2\) are both solutions of \(y'' + 10y = e^{-t}\). Then \(y_3 = y_1 + y_2\) is also a solution of the same equation.
\[
\text{False, } y_3 \text{ would solve } y'' + 10y = 2e^{-t}
\]

8. (6 points) A tank with capacity 2000 liters initially contains 1000 liters of water with 20 kg
of dissolved salt. Water containing \(0.5 + e^{-t}\) kg/liter of salt enters at a rate of 3 liters/min,
and the well-stirred mixture flows out of the tank at a rate of 2 liters/min. Let \(Q(t)\) denote
the amount of salt in the tank at any time \(t\). Compose (but do NOT solve) an initial value
problem that accurately describes \(Q(t)\), for \(0 < t < 1000\).

\[
\begin{align*}
S(t): & & S(0) &= 1000 \\
& & \frac{dS}{dt} &= 3 - 2 = 1 \quad \Rightarrow \quad S(t) = 1000 + t. \\
\end{align*}
\]

\[
\begin{align*}
Q(t) \text{ solves} & & \int \frac{dQ}{dt} &= (3)(0.5 + e^{-t}) - 2 \frac{CQ}{1000 + t} \\
& & \int & & \begin{cases} 
Q(0) = 20 \\
\end{cases} \\
& & \text{for} & & t \in (0, 1000).
\end{align*}
\]
9. (12 points) Consider the autonomous differential equation
\[ y' = (e^y - 1)(y - 3)(5 - y). \]

(a) (2 points) Find all equilibrium solutions of this equation.
\[
\begin{align*}
  e^y - 1 &= 0 \quad \Rightarrow \quad y = 0 \\
  y - 3 &= 0 \quad \Rightarrow \quad y = 3 \\
  5 - y &= 0 \quad \Rightarrow \quad y = 5 \\
\end{align*}
\]
There are the equilibrium solutions.

(b) (6 points) Classify the stability of each equilibrium solution. Justify your answer.

\[ f(y) = (e^y - 1)(y - 3)(5 - y) \]
\[
\begin{align*}
  y = 0 & \text{ is stable} \\
  y = 3 & \text{ is unstable} \\
  y = 5 & \text{ is stable} \\
\end{align*}
\]

(c) (2 points) Suppose \( y(0) = \alpha \), and \( \lim_{t \to \infty} y(t) = 0 \). Find all possible value(s) of \( \alpha \).
\[ \boxed{\alpha < 3} \]

(d) (2 points) (Circle the correct answer.) Which one of the two initial conditions below will result in a constant solution to the equation? What is the resultant solution?
\[ y(3) = 4, \quad \text{or} \quad y(-2) = 5. \]

\[ y(3) = 4 \]
\[
\Rightarrow \quad y \to 5 \quad \text{as} \quad t \to \infty
\]
\[ \therefore \text{not constant} \]

\[ y(-2) = 5 \quad \text{is an equilibrium.} \]
10. (10 points) Consider the following list of differential equations:

A. \( u'' + 2u' + 5u = 0 \)
B. \( u'' - 4u' + 13u = 0 \)
C. \( u'' + 17u = 0 \)
D. \( u'' + u = \sin(2t) \)
E. \( u'' + 4u' + 4u = \sin(2t) \)
F. \( u'' + 9u = 2 \cos(3t) \)
G. \( u'' + 6u' + 8u = 0 \)

Each of the equations above may or may not describe the displacement of a mass-spring system. Each question below has exactly one correct answer. The same equation may be reused to answer more than one question.

(a) Which equation describes a mass-spring system that is undergoing resonance?

\[ F_{\text{since}} \quad r = 3 \text{i} \]

(b) Which equation describes a mass-spring system that exhibits a simple harmonic motion?

\[ C_{\text{since}} \quad y = 0 \quad \& \quad F(t) = 0, \quad m, k > 0. \]

(c) Which equation describes a mass-spring system whose motion crosses the equilibrium position at most once?

\[ G_{\text{since}} \quad r = -2, -4 \in \mathbb{R}. \]

(d) Which equation describes a mass-spring system that is underdamped?

\[ A \]

(e) Which equation describes a mass-spring system that is overdamped?

\[ G \]
11. (10 points) Consider the differential equation

\[(2x^2 - 6y^2)y' + 4xy - 2e^{2x} = 0.\]

(a) (3 points) Verify that it is an exact equation.

\[N_x = 2x^2 - 6y^2 \Rightarrow N_x = 4x \]

\[M = 4xy - 2e^{2x} \Rightarrow M_y = 4x \]

\[= N_x \quad \text{exact.}\]

(b) (7 points) Find the solution of this equation satisfying \(y(0) = -1\). You may leave your answer in an implicit form.

\[\Psi = \int M \, dx = \int N \, dy.\]

\[\Psi = \int 2x^2 \, dy - 2e^{2x} \, dx = 2x^2y - e^{2x} + C_1(y)\]

\[\int N \, dx = \int 2x^2 - 6y^2 \, dx = 2x^2y - 2y^3 + C_2(x).\]

\[\Psi = 2x^2y - 2y^3 + C_2(x) = e^{-2x} \]

\[C_1(y) = -2y^3 \quad \text{and} \quad C_2(x) = e^{-2x}.\]

\[\Psi = 2x^2y - e^{2x} - 2y^3 = C.\]

\[(x, y) = (0, -1)\]

\[-1 - 2(-1) = C \Rightarrow C = 1.\]

So, implicit solution is 

\[2x^2y - e^{2x} - 2y^3 = 1.\]
12. (10 points) Given that \( y_1(t) = (t + 4)^3 \) is a known solution of the linear differential equation,

\[
(t + 4)^2 y'' - 5(t + 4)y' + 9y = 0, \quad t > -4.
\]

Use reduction of order to find the general solution of the equation.

\[
p(t) = \frac{-5}{t+4} \quad \Rightarrow \quad W = e^{\int p dt} = e^{\int \frac{-5}{t+4} dt} = (t+4)^5.
\]

\[
y_2 = vy_1, \quad \Rightarrow \quad v' = \frac{W}{y_1^2} = \frac{(t+4)^5}{(t+4)} = (t+4)^{-1}.
\]

So \( v = \ln(t+4) \) and hence \( y_2 = (t+4)^3 \ln(t+4). \)

So the general solution is

\[
y = C_1 (t+4)^3 + C_2 (t+4)^3 \ln(t+4).
\]
13. (14 points) Consider the second order nonhomogeneous linear equation
\[ y'' - 7y' + 12y = 10 \sin t + 12t + 5. \]

(a) (3 points) Find \( y_c(t) \), the solution of its corresponding homogeneous equation.
\[
\begin{align*}
\lambda^2 - 7\lambda + 12 &= 0 \\
(\lambda - 3)(\lambda - 4) &= 0 \\
\lambda &= 3, 4.
\end{align*}
\]
So \( y_c = C_1 e^{3t} + C_2 e^{4t} \).

(b) (7 points) Find a particular function \( Y(t) \) that satisfies the equation.
\[
\begin{align*}
Y_1 &= A \sin t + B \cos t \\
Y_1' &= A \cos t - B \sin t \\
Y_1'' &= -A \sin t - B \cos t
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow -A \sin t - B \cos t - 7(A \cos t - B \sin t) + 12(A \sin t + B \cos t) = 10 \sin t \\
&\Rightarrow -7A - 12B + 12D = 12t + 5
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow C = 1 & 12D = 12 \Rightarrow D = 1. \quad \text{So} \quad Y_2 = t + 1.
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow A + 7B + 12A = 10 \\
&-B - 7A + 12B = 0
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow 11A + 7B = 10 \\
&\Rightarrow 11B - 7A = 0 \Rightarrow B = \frac{7}{11} A \\
&\Rightarrow A = \frac{10}{11} \\
&\Rightarrow B = \frac{7}{17}
\end{align*}
\]

Particular solution: \( Y = \frac{11}{17} \sin t + \frac{7}{17} \cos t + t + 1 \).

(c) (1 point) Write down the general solution of the equation.
\[
y = y_c(t) + y_p(t) = C_1 e^{3t} + C_2 e^{4t} + \frac{11}{17} \sin t + \frac{7}{17} \cos t + t + 1.
\]

(d) (3 points) What is the form of particular solution \( Y \) that you would use to solve the following equation using the Method of Undetermined Coefficients? DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.
\[
y'' - 7y' + 12y = 3t^2 e^{4t} - t e^{3t} \sin t.
\]
\[
Y = \left( A t^2 + B t + C \right) t e^{4t} + \left( D t + E \right) e^{3t} \sin t + \left( F t + G \right) e^{3t} \cos t.
\]