

1. (i) Linear, homogeneous, order 2.
 (ii) Nonlinear, order 1.
 (iii) Linear, nonhomogeneous, order 3.

2. (a)

$$y'' - 4y' + 13y = 0$$

$$\lambda^2 - 4\lambda + 13 = 0$$

$$\lambda_{1,2} = 2 \pm 3i$$

$$y(t) = C_1 e^{2t} \cos 3t + C_2 e^{2t} \sin 3t$$

(b)

$$y'' + 2y' + y = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$\lambda_{1,2} = -1$$

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}$$

3. (a)

$$y' = \frac{\cos t + 1}{y}, \quad y(0) = -3$$

$$y dy = (\cos t + 1) dt$$

$$\frac{y^2}{2} = \sin t + t + \text{const}$$

$$\frac{(-3)^2}{2} = \sin 0 + 0 + \text{const}$$

$$\text{const} = \frac{9}{2}$$

$$y(t) = +\sqrt{2 \left(\sin t + t + \frac{9}{2} \right)}$$

(b)

$$ty' + 3y = 3, \quad y(1) = 2$$

$$y' + \frac{3}{t}y = \frac{3}{t}$$

$$\mu(t) e^{\int \frac{3}{t} dt} = e^{3 \int \frac{1}{t} dt} = t^3$$

$$y(t) = \frac{\int t^3 \frac{3}{t} dt}{t^3} = \frac{t^3 + \text{const}}{t^3} = 1 + \frac{\text{const}}{t^3}$$

$$2 = 1 + \frac{\text{const}}{1^3}$$

$$\text{const} = 1$$

$$y(t) = 1 + \frac{1}{t^3}$$

4. (a)

$$D(0) = 1000, \quad \frac{dD}{dt} = \frac{1}{10}D(t) - 120$$

(b)

$$D(t) = \frac{120}{\frac{1}{10}} + Ce^{\frac{1}{10}t}$$

$$1000 = 1200 + Ce^0$$

$$C = -200$$

$$\boxed{D(t) = 1200 - 200e^{\frac{1}{10}t}}$$

(c)

$$D(t) = 0$$

$$1200 - 200e^{\frac{1}{10}t} = 0$$

$$e^{\frac{1}{10}t} = 6$$

$$\boxed{t = 10 \ln 6}$$

5. (a) Both $\ln |t - 10|$ and $\frac{1}{t^2 - t}$ must be continuous, that is $t - 10 \neq 0$ and $t^2 - 2 \neq 0$ and therefore we have 3 "holes" in the real line: 0, 1, and 10. The maximal interval without holes, containing 8, is (1, 10).

(b) Both the function and its partial derivative with respect to y must be continuous:

$$(1 - t^2 - y^2)^{\frac{1}{2}} \quad \text{and} \quad \frac{1}{2}(1 - t^2 - y^2)^{-\frac{1}{2}}(-2y)$$

This means $1 - t^2 - y^2 > 0$, that is $t_0^2 + y_0^2 < 1$ (point (t_0, y_0) must lie within the unit circle about the origin).

6. (a) The critical points are $y = -3, 0, 3$.

(b) The sign of the right-hand side of the equation changes according to the following pattern as y goes from $-\infty$ to $+\infty$: $+ - + -$. Hence -3 is stable, 0 is unstable, 3 is stable.

(c) If $y(0) = -2$ then the solution starts between $y = -3$ and $y = 0$, and therefore y' is negative and the solution must decrease to the lower equilibrium solution:

$$\boxed{\lim_{t \rightarrow \infty} y(t) = -3}$$

7. (a)

$$\frac{\partial}{\partial y}(e^x \sin y + x^2) = \frac{\partial}{\partial x}(e^x \cos y + y) = e^x \cos y$$

(b)

$$\psi = \int (e^x \sin y + x^2) dx = e^x \sin y + \frac{1}{3}x^3 + C(y)$$

$$\frac{\partial}{\partial y}(e^x \sin y + \frac{1}{3}x^3 + C(y)) = e^x \cos y + C'(y) = e^x \cos y + y$$

$$C'(y) = y$$

$$C(y) = \frac{1}{2}y^2 + \text{const}$$

$$\boxed{e^x \sin y + \frac{1}{3}x^3 + \frac{1}{2}y^2 = \text{const}}$$

8. (a)

$$y'' + y' - 2y = 0$$

$$\lambda^2 + \lambda - 2 = 0$$

$$\lambda_{1,2} = 1, -2$$

$$\boxed{y(t) = C_1 e^t + C_2 e^{-2t}}$$

(b) First, $\lim_{t \rightarrow \infty} y(t) = 0$ means $C_1 = 0$.

$$y(0) = C_1 + C_2 = C_2 = \beta$$

$$y'(0) = C_1 - 2C_2 = -2C_2 = 1$$

$$\boxed{\beta = C_2 = -\frac{1}{2}}$$

9. (a) Plug the functions into equation and see what happens.

(b)

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = -4t^{-1} \neq 0$$

Yes, they are the fundamental set of solutions.