Math 251
February 24, 2005 First Exam

NAME:______________________________ Section #:__________

There are 9 questions on this exam. Question 9 is worth 12 points. Each other question is worth 11 points. The points assigned to each part of the question are indicated at the start of the part.

Show all your work. Partial credit may be given.

The use of calculators, books, or notes is not permitted on this exam.

Please turn off your cell phone before starting this exam.

Time limit 1 hour and 15 minutes.

1. a. 9pts Find the general solution to the following ODE

$$ty' + 2y = 2 \qquad t > 0$$

This is a linear equation.
2pts It must be put into the following form:

$$y' + \frac{2}{t}y = \frac{2}{t}$$

3pts $p = \frac{2}{t}$ $\int \frac{2}{t}\,dt = 2\ln t = \ln t^2$
The integrating factor is $\mu = e^{\int p\,dt} = t^2$

3pts Multiplying through by the integrating factor gives $(yt^2)' = 2t$

1pts Integrating gives $yt^2 = t^2 + C$

Finally $y = 1 + Ct^{-2}$
(Do not remove points of leaving in implicit form)

b. 2pts Find the solution of the above equation which satisfies $y(1) = 0$.
2pts Plug in $t = 1$ and $y = 0$ to find $C = -1$

2. **a. 3pts** Verify that the following ODE is exact:

$$2t + e^y + (te^y - \cos y)\frac{dy}{dt} = 0$$

(Show your work.)

1pts Identify: $F_t = 2t + e^y$ and $F_y = te^y - \cos y$

2pts Check $F_{ty} = e^y$ and $F_{yt} = e^y$

b. 6pts Find the general solution to the ODE in Part **a.**
(You may leave your answer in implicit form.)

2pts We see that $F = t^2 + e^y + h(y)$.

2pts Differentiating this with respect to y : $F_y = 0 + e^y + \frac{dh}{dy}$ and comparing gives $\frac{dh}{dy} = -\cos y$

2pts We see that $F = t^2 + e^y - \sin y$.

1pts The form of the general solution is $F(t, y) = C$.

The answer $F = t^2 + e^y - \sin y + C$ does not indicate awareness of the form of the general solution. Please take off 1pt for this.

c. 2pts Find the solution to the ODE in Part **a.** which satisfies $y(1) = \pi/2$. (You may leave your answer in implicit form.)

2pts Plugin $y = \pi/2$ and $t = 1$ to obtain $C = e^{\pi/2}$ The solution is given by the equation: $t^2 + e^y - \sin y = e^{\pi/2}$.

3. 11pts Solve the following ODE.

$$y'' = y(y')^3$$

(You may leave your answer in implicit form.)

This is the case of the missing t. The strategy for solving is to introduce a new unknown function $v = y'$ and view y being the independent variable temporarily.

4pts We see that $y'' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$

The ODE is now $\frac{dv}{dy}v = yv^3$

2pts $y =$const solves the equation. So we assume that v is not zero (the function) and the ODE is now $v^{-2}\frac{dv}{dy} = y$

2pts We integrate both sides with respect to y:

$\int v^{-2}\frac{dv}{dy}\,dy = \int y\,dy$

$-v^{-1} = \frac{1}{2}y^2 + C$

3pts We now return to viewing t as the independent variable: $-2 = (y^2 + C)y'$ This is easy to integrate with respect to t: $-2t + D = \frac{1}{3}y^3 + Cy$

Take off 1pt for omitting either one of C or D and 2pts for omitting both.

Take off 1pt for not mentioning the solution y=const.

4. **a.** **6pts** A ski slope operator determines that, without producing artificial snow, the volume of snow on the ski slope decreases at a rate equal to $-1/10$ of the volume present. If the operator produces artificial snow at the rate of 10^6 cubic meters per day, then write a differential equation for the volume of snow.
(DO NOT SOLVE the ODE.)

2pts Let V be the volume of snow at time t.

4pts $V' = \frac{-1}{10}V + 10^6$

Take off 2pts for getting either sign wrong.
Take off 2pts for writing $10^6 t$ instead of 10^6

b. **5pts** Suppose that after 10 days of operation there are 5x10^6 cubic meters of snow on the slopes. Use Euler's method with one step to approximate the amount of snow on 11th day.

1pts The choice of t_0 does not influence the answer to this question since the ODE is autonomous. However, for the sake of definiteness take $t_0 = 10$.
Then $V_0 = 5 \times 10^6$ and $h = 1$.

2pts Now $V'(t_0, V_0) = \frac{-1}{10} 5 \times 10^6 + 10^6 = \frac{1}{2} \times 10^6$

2pts Therefore $V_1 = V_0 + V'(t_0, V_0)h = \frac{11}{2} \times 10^6$

5. 11pts Consider the ODE

$$t^2y'' - ty' + y = 0$$

Given that $y_1 = t$ is a solution, find another solution y_2 of this ODE that is not a constant multiple of y_1.

4pts We seek $y_2 = vy_1 = vt$. We plug into the given ODE and find a new ODE for v:

$y_2 = vt$
$y_2' = v't + v$
$y_2'' = v''t + 2v'$

3pts The ODE for v is $v''t^3 + v't^2 = 0$
Or, $v''t + v' = 0$
We see that $\frac{v''}{v'} = -\frac{1}{t}$

2pts Integrating with respect to t:

We see that $\ln v' = -\ln t$

2pts Exponentiating gives: $v' = \frac{1}{t}$
and hence $v = \ln t$
So $y_2 = t \ln t$

6. **a. 4pts** Find the general solution of $y'' + 6y' + 10y = 0$

ANS.

2pts The characteristic polynomial $r^2 + 6r + 10$ has complex roots $r = -3 \pm i$

2pts Therefore $y = e^t(c_1 \cos t + c_2 \sin t)$.

b. 3pts Find the general solution of $y'' + 6y' + 9y = 0$

ANS.

1pts The characteristic polynomial $r^2 + 6r + 9$ has double root $r = -3$

2pts Therefore $y = e^{-3t}(c_1 + c_2 t)$.

c. 2pts Solve the initial value problem:

$$y'' + 6y' + 9y = 0 \qquad y(0) = 1, \quad y'(0) = 0$$

2pts Setting $t = 0$ in the above gives $c_1 = 1$ and $c_2 = 3$. Therefore $y' = -3y + e^{-3t}c_2$.

d. 2pts Find $\lim_{t\to\infty} y(t)$, where $y(t)$ is the solution found in Part **c.**

2pts By L'Hospital's rule, the limit is zero.

7. Consider the differential equation

$$y' = -y^2 + 4y - 3$$

a. 3pts Determine all the equilibrium solutions of this ODE.

$$-y^2 + 4y - 3 = -(y^2 - 4y + 3) = -(y - 3)(y - 1)$$

$y = 3$ and $y = 1$ are the equilibrium solutions.

b. 4pts Sketch a direction field for this ODE.

Use the Direction Field applet to sketch the direction field.

c. 4pts For each equilibrium solution found in Part **a.** determine whether it is **asymptotically stable** or **unstable.**

2pts $y = 3$ is asymptotically stable.

2pts $y = 1$ is unstable

8. **a. 7pts** Let y_1, y_2 be two solutions to the equation $ty" - 2y' - y = 0$. Determine the Wronskian $W(y_1, y_2)$ of y_1 and y_2.

1pts Rewrite the ODE: $y" - \frac{2}{t}y' - \frac{1}{t}y = 0$.

6pts According to Abel's formula: $W(y_1, y_2) = ce^{-\int p\,dt} = ce^{(-1)(-2\ln t)} = ct^2$

Take off two points for getting not remembering the minus sign in the formula.
Take off two points for getting not remembering the constant c in the formula.

b. 4pts If $W(y_1, y_2)(2) = 1$, thendetermine $W(y_1, y_2)(3)$.

2pts Plug in $W(y_1, y_2)(2) = 1$ to obtain $4c = 1$, $c = 1/4$.

2pts So $W(y_1, y_2)(3) = (1/4)3^2 = 9/4$.

9. **a. 3pts** Consider the ODE $(t-1)y' + 3y = 2$. Determine all pairs (t_0, y_0) for which the uniqueness of the solution to the IVP $y(t_0) = y_0$ is **NOT** guaranteed?

ANS.

1pts Rewrite equation $y' + \frac{3}{t-1} y = \frac{2}{t-1}$.

2pts This is a linear ODE. The coefficients are discontinuous at $t_0 = 1$. So uniqueness is **NOT** guaranteed when for all (t_0, y_0) with $t_0 = 1$.

b. 3pts Consider the ODE $(t-1)y' + y^{1/3} = 2$. Determine all pairs (t_0, y_0) for which the uniqueness of the solution to the IVP $y(t_0) = y_0$ is **NOT** guaranteed?

ANS.

1pts Rewrite equation $y' = f(t, y) = -\frac{1}{t-1} y^{1/3} + \frac{2}{t-1}$.

This is a nonlinear ODE. The $f(t, y)$ discontinuous at $t_0 = 1$. The partial with respect to y, f_y is discontinuous if $t_0 = 1$ or if $y_0 = 0$.

2pts So uniqueness is **NOT** guaranteed when for all (t_0, y_0) with $t_0 = 1$ or $y_0 = 1$.

c. 3pts Which one of the following is a second order ODE? Circle it.

$$y^2 = \cos^2 t \qquad y(y')^2 = \sin t \qquad \frac{y''}{y} = \tan t$$

3pts The last equation is second order

d. 3pts Which one of the following is a linear ODE? Circle it.

$$t^4 y' + t^3 y = \sin^2 t \qquad 1 = y\,y' \qquad t = y' e^y$$

3pts The first equation is linear.