Math 251H Second Midterm Exam

75 minutes

March 25, 2009

SAMPLE EXAM.

1. A 3 kg mass is suspended from a spring, which stretches the spring 5 m from its natural length. The system is placed into the liquid with damping constant 14 Newton-seconds per meter. At t = 0, the system is at rest at its equilibrium position, then receives an external force of $6\cos(\omega t)$ Netwons. Assume ω is positive and $g = 10m/s^2$.
(a) Set up an initial value problem that describes the motion of the mass. Be sure to explain any variables that appear in your equation.
(b) For which value of ω will the system have resonance?
(c) Find the steady-state solution.
Solution: m= 3. From the equilibrium condition $mg = KL$, we have, $K = 6$. In addition, $\gamma = 14$. Therefore, the equation of motion is given by,

$$3u'' + 14u' + 6u = 6\cos\omega t, u(0) = 0, u'(0) = 0.$$

The natural frequency is $\sqrt{K/m} = \sqrt{2}$. Resonance occurs when the frequency is close to the natural frequency.

For the steady state solution, let,

$$u(t) = A\cos\omega t + B\sin\omega t.$$

A substitution yields the following two equations,

$$\begin{aligned}(6-3\omega^2)A + 14\omega B &= 6\\ (6-3\omega^2)B - 14\omega A &= 0.\end{aligned}$$

We find, $B = 84\omega/(36+160\omega^2+9\omega^4)$, and $A = 6(6-3\omega^2)/(36+160\omega^2+9\omega^4)$.

2.

(a) Find the Laplace transform of

$$f(t) = \begin{cases} 9, & 0 \le t < 3, \\ t^2, & t \ge 3. \end{cases}$$

Solution.

$$f(t) = u_3(t)t^2 + (1 - u_3(t))9$$

. We can write, $t^2 = (t + 3 - 3)^2$. Therefore,

$$F(s) = e^{-3s}\mathcal{L}(t+3)^2 + 9/s - e^{-3s}/s = e^{-3s}(2/s^3 + 6/s^2 + 9/s) + 9/s - e^{-3s}/s$$

(b) Find the Laplace transform of

$$f(t) = e^t \sin(\sqrt{2}\, t).$$

Solution. Let $g(t) = \sin\sqrt{2}t$. So $G(s) = \frac{\sqrt{2}}{s^2+2}$. Therefore,

$$F(s) = G(s-1) = \frac{\sqrt{2}}{s^2 - 2s + 3}.$$

(c) Find the inverse Laplace transform of

$$F(s) = \frac{s-2}{s^2 + 2s + 10}.$$

Solution.

$$f(t) = e^{-t}\Big(\cos 3t - \sin 3t\Big).$$

3. Solve the following initial value problem using Laplace transform,

$$y^{(4)} - 16y = 0, y(0) = 1, y'(0) = 2, y''(0) = 0, y^{(3)}(0) = 0.$$

Solution. Taking Laplace transform, we find,

$$Y(s) = \frac{s^3}{s^4 - 16} = \frac{As + B}{s^2 + 4} + \frac{C}{s - 2} + \frac{D}{s + 2}.$$

$$A = 1/2, B = 0, C = 1/4, D = 1/4.$$

and

$$y(t) = \frac{1}{2}\cos 2t + \frac{1}{4}(e^{2t} + e^{-2t}).$$

4. Solve the following initial value problem,

$$y'' + 6y' + 13y = g(t),$$

where

$$g(t) = \begin{cases} t-1, & 0 \le t < 1, \\ 0, & t \ge 1. \end{cases}$$

sketch the solution in time.

Solution. Assume the initial condition is zero. First we have,

$$g(t) = (1 - u_1(t))(t-1).$$

So,

$$G(s) = 1/s^2 - 1/s - e^{-s}/s^2.$$

For the solution, we have,

$$Y(s) = \frac{1}{s^2+6s+13} G(s).$$

By doing partial fractions, we have,

$$\frac{1}{(s^2+6s+13)s} = -\frac{1}{13}\frac{s+6}{s^2+6s+13} + \frac{1}{13}\frac{1}{s},$$

and,

$$\frac{1}{(s^2+6s+13)s} = \frac{1}{13} 1s^2 + \frac{1}{169}\frac{6s+23}{s^2+6s+13} - \frac{6}{169}\frac{1}{s}.$$

The final answer is,

$$y(t) = -19/169 + 1/13t + \frac{1}{169} e^{-3t}\Big(19\cos 2t + 22\sin 2t\Big)$$

$$+u_1(t)\Big(19/169 - 1/13t - \frac{1}{338} e^{-3t+3}(12\cos(2t-2) + 5\sin(2t-2).\Big)$$

5. Solve the following initial value problem.

$$y'' - 2y = 4e^{-2t}\delta(t-1), y(0) = 0, y'(0) = 1.$$

Sketch the solution.

The solution

$$y(t) = -1/2 + 1/2\, e^{2t} - 2u_1(t)(-e^2 + e^{2t}).$$