Math 251H-01

Midterm Exam 1

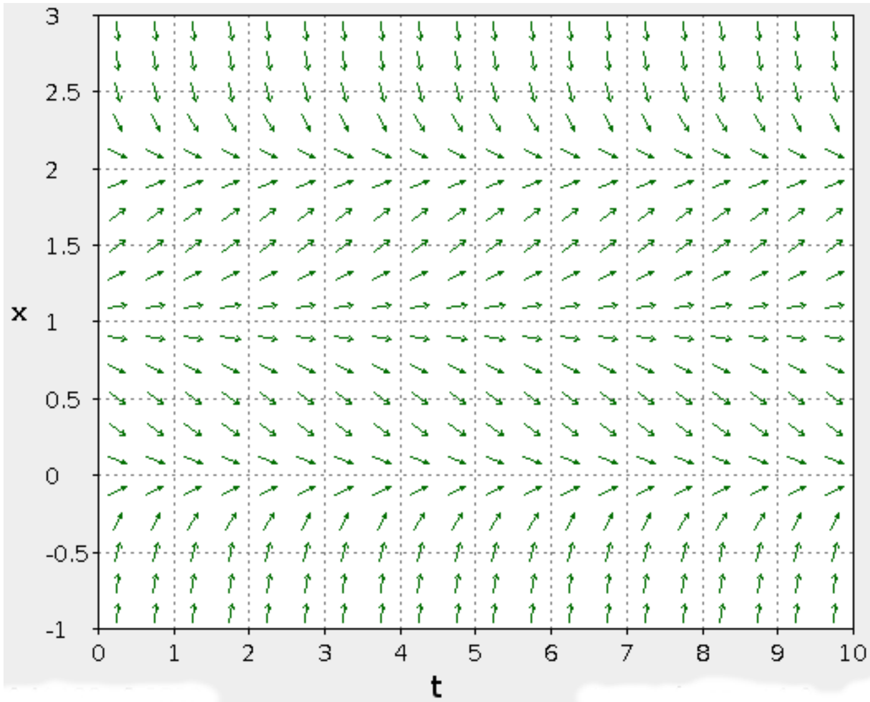
February 25, 2016

Name:____

Instructions: Clearly answer each of the questions. Box your answers. Partial credit will be awarded based on the clarity and correctness of your explanation of each solution. Use the backs of pages for your scratch work. There are **9** problems. Make sure you have all **11** pages.

Problem	**out of**	**score**
1	5	
2	6	
3	5	
4	8	
5	8	
6	8	
7	6	
8	6	
9	8	
Total	60	

1. (**5** points) Consider the direction field



(a) What is the differential equation associated with this direction field? (circle the correct one)

(i) $x'(x) = x(x+1)(x+2)$ (ii) $x'(t) = x(1-x)(x-2)$

(iii) $x'(t) = -x(x+1)(x+2)$ (iv) $x(t) = x(x-1)(x-2)$

(b) On the direction field, sketch & label an approximate solution if $x(0) = 0.5$.
(c) On the direction field, sketch & label an approximate solution if $x(0) = 1.5$.
(d) List all asymptotically stable solutions, if any (you do not need to justify your answer).
Solution: $x = 0$ and $x = 2$ are the asymptotically stable solutions.

(e) List all unstable solutions, if any (you do not need to justify your answer).
Solution: $x = 1$ is the unstable solution.

2. (**6** points) Solve the initial value problem

$$\begin{cases} \sqrt{y}\,dx + (1+x)\,dy = 0, \quad -1 < x < e^2 - 1 \\ y(0) = 1 \end{cases}$$

Solution: Rewrite first order ODE as

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{1+x}.$$

Note that it's separable!

$$\begin{aligned} \int \frac{dy}{\sqrt{y}} &= -\int \frac{dx}{1+x} \\ 2\sqrt{y} &= \ln\left(\frac{1}{1+x}\right) + C \\ \sqrt{y} &= \left(\ln\left(\frac{1}{\sqrt{1+x}}\right) + \frac{C}{2}\right). \end{aligned}$$

Since $y(0) = 1$, $C = 2$, and

$$\sqrt{y} = \left(\ln\left(\frac{1}{\sqrt{1+x}}\right) + 1\right).$$

Since since K arbitrary. Note that $x > -1$ so there are no problems with the ln or the $\surd$, and $x < e^2 - 1$, so the expression on the right hand side is positive.

The solution to the IVP on $-1 < x < e^2 - 1$ is

$$y(x) = \left(1 + \ln\left(\frac{1}{\sqrt{1+x}}\right)\right)^2$$

3. (**5** points) Find the most general function $M(x,y)$ so that the equation

$$M(x,y)\,dx + (\sin x \cos y - xy - e^y)\;dy = 0$$

is exact.

Solution: Let the solution be $F(x,y)$. Then

$$\frac{\partial F}{\partial y} = (\sin x \cos y - xy - e^y) \Rightarrow F(x,y) = \sin x \sin y - \frac{xy^2}{2} - e^y + g(x),$$

where g is some arbitrary function of x. Then $M(x,y)$ must be

$$M(x,y) = \frac{\partial F}{\partial x} = \cos x \sin y - \frac{y^2}{2} + g'(x).$$

But $g'(x)$ is just an arbitrary function of x, so can just as easily call it $f(x)$.
Therefore the most general function $M(x,y)$, so that the equation is exact, is

$$M(x,y) = \cos x \sin y - \frac{y^2}{2} + f(x).$$

Note: Could also **differentiate** $N(x,y) = (\sin x \cos y - xy - e^y)$ with respect to x, set that equal to $\frac{\partial M}{\partial y}$, and integrate. Would get to the same place!

4. (**8** points) A brine solution of salt flows at a constant rate of 4 L/min into a large tank that initially held 100 L of pure water. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 3 L/min. Assume that the concentration of salt in the brine entering the tank is 0.2 kg/L.

(a) Determine the **concentration** of salt in the tank after t min.

(b) What is the limiting concentration, i.e. as $t \to \infty$? Does your answer make sense?

Solution: (a) First determine the volume in the tank at time t, $V(t)$. Initial value problem for $V(t)$

$$\begin{cases} \frac{dV}{dt} = 1, & t \geq 0 \\ V(0) = 100 \end{cases}$$

has solution $V(t) = 100 + t$ L.

Next determine the mass at time t, $x(t)$. Initial value problem for $x(t)$:

$$\begin{cases} \frac{dx}{dt} = 0.8 - \frac{3x}{100+t}, & t \geq 0 \\ x(0) = 0 \end{cases}$$

We can solve the ODE

$$\frac{dx}{dt} + \frac{3x}{100+t} = 0.8$$

using an integrating factor $\mu(t) = \exp\left(\int \frac{3dt}{100+t}\right) = (100+t)^3$. Then

$$\frac{d}{dt}\left((100+t)^3 x\right) = 0.8(100+t)^3$$

Integrating both sides,

$$\begin{aligned} (100+t)^3 x &= 0.2\,(100+t)^4 + C \\ \Rightarrow x(t) &= 0.2(100+t) + \frac{C}{(100+t)^3}. \end{aligned}$$

Since $x(0) = 0$, $C = -2 \times 10^7$, and the mass at time t is

$$x(t) = 0.2(100+t) - \frac{2 \times 10^7}{(100+t)^3}.$$

But we're asked for the concentration! The concentration at time t, $C(t) = x(t)/V(t)$.

The concentration at time t is

$$C(t) = 0.2 - \frac{2 \times 10^7}{(100+t)^4}.$$

(b) The limiting concentration is 0.2 kg/L. Makes sense: this is the concentration of the solution flowing in!

5. (**8** points) The function $y_1(x) = x$ solves the differential equation

$$x^2y'' - xy' + y = 0$$

on the interval $x > 0$.

(a) Find a second linearly independent solution $y_2(x)$.

(b) Prove that $y_1(x)$ and your $y_2(x)$ are indeed linearly independent.

(c) Write down the general solution of $x^2y'' - xy' + y = 0$.

Solution: (a) Use reduction of order. Let the second solution $y_2(x) = v(x)y_1(x) = xv(x)$. Then $y_2' = v + xv'$, $y_2'' = 2v' + xv''$, and our equation becomes

$$\begin{aligned} x^2\left(2v' + xv''\right) - x\left(v + xv'\right) + vx &= 0 \\ x^3v'' + x^2v' &= 0 \end{aligned}$$

Let $w(x) = v'(x)$. In standard form, the equation for w is $w'' + \frac{1}{x}w' = 0$. Solve using an integrating factor, $\mu(x) = \exp\left(\int \frac{dx}{x}\right) = x$. Then

$$\begin{aligned} xw' + w &= 0 \\ \Rightarrow \frac{d}{dx}(xw) &= 0 \\ w &= \frac{C}{x}. \end{aligned}$$

But $w = v'$ so

$$v' = \frac{C}{x} \Rightarrow v = C\ln x + D.$$

For $y_2(x)$ take $x-$dependent part only. $y_2(x) = vy_1 = x\ln x$.

Thus the second linearly independent solution is $y_2(x) = x\ln x$.

(b) Use the Wronskian to show linear independence.

$$W[y_1, y_2] = y_1y_2' - y_1'y_2 = (x)(\ln x + 1) - (1)(x\ln x) = x.$$

Our equation is defined on $x > 0$. $W[y_1, y_2] = x \neq 0$ on $x > 0$. Therefore $y_1 = x$ and $y_2 = x\ln x$ are linearly independent.

(c) The general solution is

$$y(x) = C_1x + C_2x\ln x.$$

6. (**8** points) Find the general solution of

$$x'' + 4x' + 4x = \frac{e^{-2t}}{t}, \ \ t > 0$$

Solution: First find the homogeneous solution, $x_h(t)$ satisfying $x_h'' + 4x_h' + 4x_h = 0$.
Let $x = e^{rt}$. The characteristic equation is $r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$ which has the repeated root $r = -2$. Therefore $x_1 = e^{-2t}$ and $x_2 = te^{-2t}$ satisfy the equation and the homogeneous solution is $x_h(t) = C_1 e^{-2t} + C_2 te^{-2t}$.
Now find the particular solution. Use variation of parameters. Let $x_p = u_1 x_1 + u_2 x_2$. Then

$$u_1(t) = -\int \frac{x_2 g}{W}\, dt \quad \text{and} \quad u_2(t) = \int \frac{x_1 g}{W}\, dt$$

where g is the inhomogeneity, $g(t) = e^{-2t}/t$, x_1 and x_2 are homogeneous solutions defined above, and W is the Wronskian

$$W = x_1 x_2' - x_1' x_2 = \left(e^{-2t}\right)\left(e^{-2t} - 2te^{-2t}\right) - \left(-2e^{-2t}\right)\left(te^{-2t}\right) = e^{-4t}.$$

Then

$$u_1(t) = -\int \frac{\left(te^{-2t}\right)\left(e^{-2t}/t\right)}{\left(e^{-4t}\right)}\, dt = -\int dt = -t$$

and

$$u_2(t) = \int \frac{\left(e^{-2t}\right)\left(e^{-2t}/t\right)}{\left(e^{-4t}\right)}\, dt = \int \frac{dt}{t} = \ln t.$$

The particular solution is $x_p = u_1 x_1 + u_2 x_2 \Rightarrow$

$$x_p = te^{-2t}\left(\ln t - 1\right).$$

Therefore the general solution is

$$x(t) = C_1 e^{-2t} + C_2 te^{-2t} + t \ln t e^{-2t}$$

(part of the particular solution is absorbed in the homogeneous solution).

7. (**6** points) Find the fundamental solution set of

$$y^{(5)} + 16y' = 0.$$

Solution: Let $y = e^{rt}$. The characteristic equation is

$$r^5 + 16r = 0$$

which has roots $r = 0$ and

$$\begin{aligned} r^4 &= -16 \\ &= 16e^{\pi i + 2n\pi i} \\ \Rightarrow r &= 2e^{\pi i/4 + n\pi i/2}. \end{aligned}$$

That is, the remaining roots are

$$r = \begin{cases} 2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2}\,(1+i), & n = 0 \\ 2\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right) = \sqrt{2}\,(1-i), & n = 3 \\ 2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2}\,(-1+i), & n = 1 \\ 2\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right) = \sqrt{2}\,(-1-i), & n = 2 \end{cases}$$

Therefore $e^{\sqrt{2}t}\cos(\sqrt{2}t)$, $e^{\sqrt{2}t}\sin(\sqrt{2}t)$, $e^{-\sqrt{2}t}\cos(\sqrt{2}t)$, $e^{-\sqrt{2}t}\sin(\sqrt{2}t)$, in addition to the constant solution indicated by the $r = 0$ root.

The fundamental solution set is

$$\left\{1, e^{\sqrt{2}t}\cos(\sqrt{2}t), e^{\sqrt{2}t}\sin(\sqrt{2}t), e^{-\sqrt{2}t}\cos(\sqrt{2}t), e^{-\sqrt{2}t}\sin(\sqrt{2}t)\right\}.$$

8. (**6** points) I've set up an LCR circuit as a crystal radio with a capacitor of 0.2×10^{-6} F and a resistor of $5 \times 10^3\,\Omega$. I want to use my radio to pick up a frequency of 1000 kHz (i.e., the forcing term is $F\cos(1000000t)$). What value should I choose for the inductor to pick up the radio signal most clearly? Why?

Solution: The dynamics on an LCR circuit are given by the ODE

$$LQ'' + RQ' + \frac{1}{C}Q = F\cos\beta t \Rightarrow LQ'' + 5\times 10^3 Q' + 5\times 10^6 Q = F\cos(1000000t).$$

To pick up the signal most clearly we want to exploit **resonance**, which occurs when the natural frequency is equal to the forcing frequency. The natural freqency is

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2\times 10^{-6}L}},$$

and the forcing frequency is

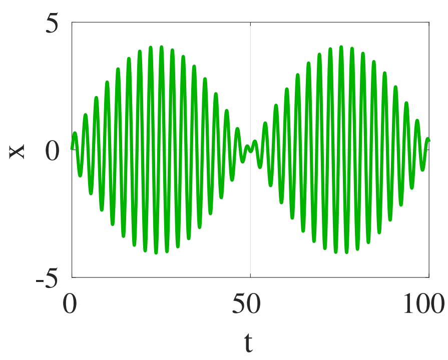
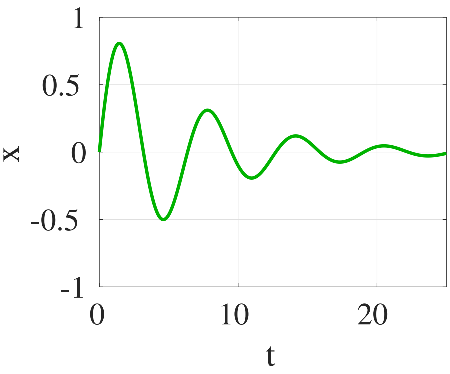
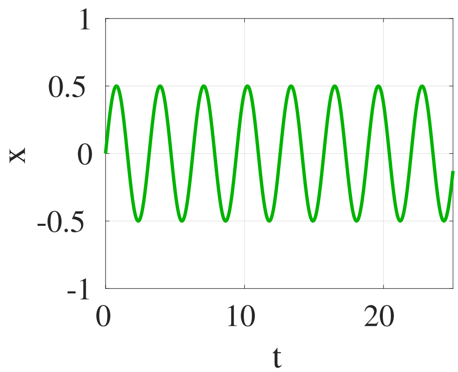
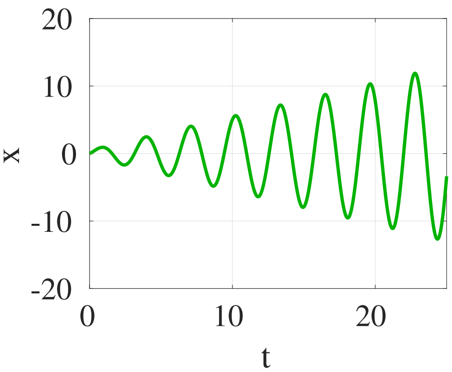
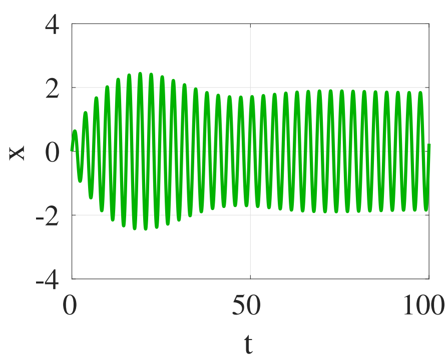
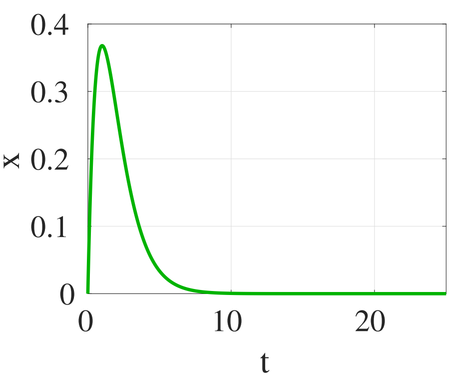
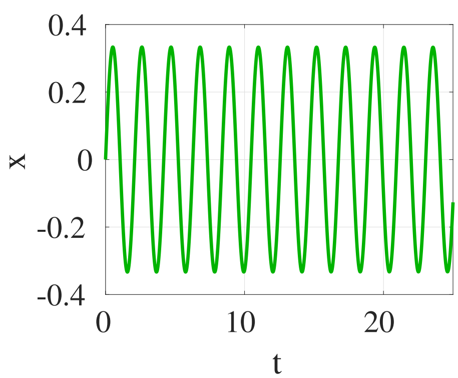
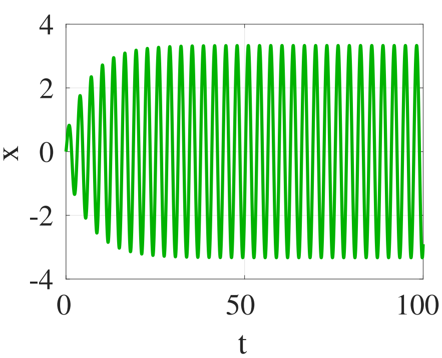
$$\beta = 1000000.$$

Setting these equal to each other,

$$\begin{aligned}
\frac{1}{\sqrt{0.2\times 10^{-6}L}} &= 1\times 10^6\\
\sqrt{\frac{5\times 10^6}{L}} &= 1\times 10^6\\
\frac{5\times 10^6}{L} &= 1\times 10^{12}\\
\frac{5}{L} &= 1\times 10^6\\
\Rightarrow L &= \frac{5}{10^6}\\
\Rightarrow L &= 5\times 10^{-6}.
\end{aligned}$$

We exploit resonance to pick up the radio signal most clearly by setting the inductor to 5×10^{-6} H.

9. (**8** points) See following page for the problem.

D		A	
G		F	
H		E	
C		B	

Consider the forced mass-spring system, which can be described by:

$$x''(t) + bx'(t) + kx(t) = F\cos(\beta t), \quad x(0) = 0, \quad x'(0) = 1,$$

where $x(t)$ is the distance of the mass from equilibrium at time t, b the damping, and k the spring constant. The forcing has amplitude F and frequency β. Assume the mass is 1 kg.
On the previous page you will find plots of different solutions $x(t)$ for various values of b, k, F, and β, corresponding in no particular order to:

- A $b = 0.3$, $k = 1$, $F = 0$, β not relevant $\Rightarrow$
  $x''(t) + 0.3x'(t) + x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$
- B $b = 0.3$, $k = 4$, $F = 2$, $\beta = 2 \Rightarrow$
  $x''(t) + 0.3x'(t) + 4x(t) = 2\cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$
- C $b = 0$, $k = 9$, $F = 0$, β not relevant $\Rightarrow$
  $x''(t) + 9x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$
- D $b = 0$, $k = 4.5$, $F = 1$, $\beta = 2 \Rightarrow$
  $x''(t) + 4.5x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$
- E $b = 2$, $k = 1$, $F = 0$, β not relevant $\Rightarrow$
  $x''(t) + 2x'(t) + x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$
- F $b = 0$, $k = 4$, $F = 2$, $\beta = 2 \Rightarrow$
  $x''(t) + 4x(t) = 2\cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$
- G $b = 0$, $k = 4$, $F = 0$, β not relevant $\Rightarrow$
  $x''(t) + 4x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$
- H $b = 0.1$, $k = 4.5$, $F = 2$, $\beta = 2 \Rightarrow$
  $x''(t) + 0.1x'(t) + 4.5x(t) = 2\cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

Match the differential equation with its solution's plot by writing the appropriate letter in the box to the *left* of the plot, on the previous page.