Math 250
Summer 2009
Final Exam

NAME: ____________________

ID No: ____________________

SECTION: ____________________

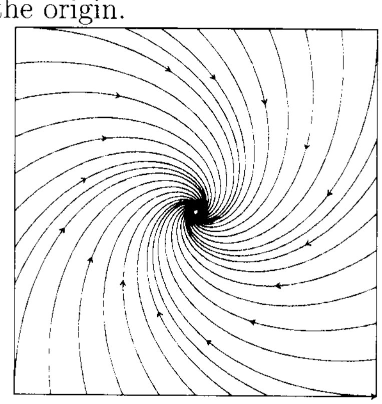
This exam contains 12 questions on 10 pages (including this title page). This exam is worth a total of 150 points. To receive full credit for a partial credit problem all work must be shown. When in doubt, fill in the details.

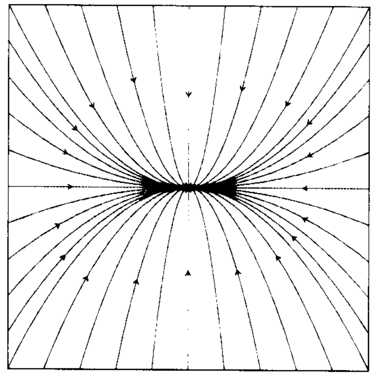
No notes, books or calculators may be used during the exam.

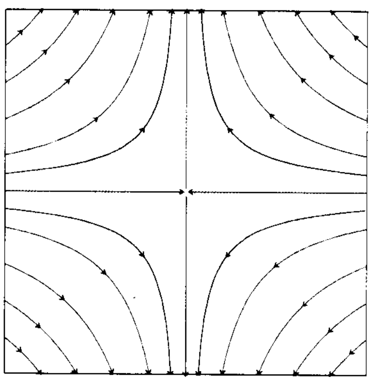
Please, Box Your Final Answer (when possible).

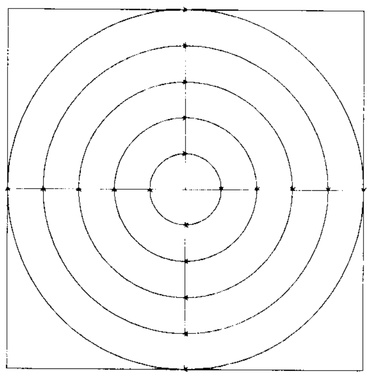
1:		12 points
2:		12 points
3:		8 points
4:		10 points
5:		10 points
6:		12 points
7:		10 points
8:		10 points
9:		6 points
10:		20 points
11:		20 points
12:		20 points
Total:		150 points

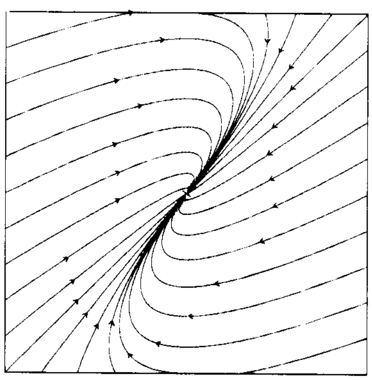
1. *(12 points)* Match the sketches of phase portraits for the 2×2 linear system $\mathbf{x}' = A\mathbf{x}$, lettered A through F, with the names of their critical points at the origin.

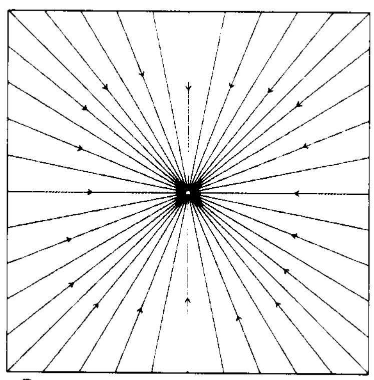
A


B


C


D


E


F


node: B

proper node(star point): F

improper node: E

spiral point: A

center: D

saddle point: C

2. *(12 points)* Match the following general solutions of 2×2 linear systems $\mathbf{x}' = A\mathbf{x}$ with the sketches of the phase portraits given in Problem 1.

E $\quad \mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} t \\ t+1 \end{bmatrix}$

F $\quad \mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

C $\quad \mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

D $\quad \mathbf{x}(t) = C_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$

A $\quad \mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$

B $\quad \mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3. *(8 points)* Rewrite the following second order linear equation as a system of two first order linear equations.

$$y'' + 2y' + y = 0.$$

$x_1 = y$

$x_2 = y'$

$$\begin{cases} x_1' = x_2 \\ x_2' + 2x_2 + x_1 = 0 \end{cases}$$

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 2x_2 \end{cases}}$$

4. *(10 points)* Find the general solution for the following differential equation.

$$y' = \frac{3x^2-1}{3+2y}.$$

$$(3+2y)y' = 3x^2-1$$

$$\int (3+2y)y'\,dx = \int 3x^2-1\,dx$$

$$\boxed{3y+y^2 = x^3-x+C}$$

5. *(10 points)* Find the general solution of the following differential equation.

$$y''+2y'+y=e^{-t}.$$

Homogeneous equation:

$y''+2y'+y=0$, $\quad r^2+2r+1=0 \Rightarrow (r-(-1))^2=0$

$r_1=r_2=-1$

General sol. of hom. eq-n: $y(t)=c_1e^{-t}+c_2te^{-t}$

Guess for a solution of non homogeneous eq-n

$Y(t)=A\,e^{-t}\cdot t\cdot t = At^2e^{-t}$

Substitute:

$(2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}) + 2(2Ate^{-t} - At^2e^{-t}) + At^2e^{-t} = e^{-t}$

e^{-t}: $2A=1 \Longrightarrow A=\frac{1}{2}$

te^{-t}: $-4A+4A=0$ ✓

t^2e^{-t}: $2A-2A=0$ ✓

$$\boxed{\begin{array}{l}\text{General solution:}\\ y(t)=C_1e^{-t}+C_2te^{-t}+\frac{1}{2}t^2e^{-t}\end{array}}$$

6. *(12 points)* Solve the following differential equation.

$$ty' + y = t, \qquad y(1) = 0.$$

$$y' + \frac{1}{t}y = 1 \qquad \Big| \cdot \mu$$

$$\mu' = \frac{1}{t}\mu \implies \boxed{\mu = t}$$

$$ty' + t'y = t$$

$$(ty)' = t \implies ty = \int t\, dt + C = \frac{1}{2}t^2 + C$$

$$y = \frac{1}{2}t + \frac{C}{t}$$

From I.C. $y(1) = \frac{1}{2} + C = 0 \implies \boxed{C = \frac{-1}{2}}$

$$\boxed{y(t) = \frac{1}{2}t + \frac{-1}{2t}}$$

7. *(10 points)* A tank initially contains 1000 gal of fresh water. Water containing 20 grams of salt per gal is entering at a rate of 5 gal per minute. A well-mixed water is is allowed to flow out of the tank at the same rate.

(a) Introduce the variables and their meaning.

$W(t)$ – ammount of water in the tank

$S(t)$ – ammount of salt in the tank

(b) Write the differential equations, and give the initial conditions, that describe this event.

$$r_{in} = r_{out} = 5 \frac{gal}{min}$$

$$\begin{cases} W'(t) = r_{in} - r_{out} = 5 - 5 = 0 \\ W(0) = 1000 \text{ gal} \end{cases}$$

$$W(t) \equiv 1000 \text{ gal}$$

$$c_{in} = 20 \frac{gr.}{gal}$$

$$S'(t) = r_{in} c_{in} - r_{out} \frac{S(t)}{W(t)} =$$

$$= 5 \cdot 20 - 5 \frac{S(t)}{1000}$$

$$S(0) = 0 \qquad \text{(fresh water)}$$

$$\boxed{\begin{cases} S'(t) = 100 - .005 \cdot S(t) \\ S(0) = 0 \end{cases}}$$

8. *(10 points)* A mass 200 g stretches a spring 10 cm. It is attached to a viscous dumper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 10 cm below its equilibrium position and given an initial downward velocity of 10 m/s, **write the differential equation with initial conditions that describes the motion of the body. Do not solve.**

Taking $g \approx 10 \frac{m}{s^2}$

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$m = .2$ Kg

$K = \frac{w}{L} = \frac{m \cdot g}{L} = \frac{.2 \cdot 10}{.1} = 20$

$\gamma = \frac{3}{5}$

$+$

$m u'' + \gamma u' + k u = 0$

$$\begin{cases} .2 u'' + \frac{3}{5} u' + 20 u = 0 \\ u(0) = .1 \\ u'(0) = 10 \end{cases}$$

9. *(6 points)* The motion of the spring mass system is governed by the following differential equation

$$y'' + \gamma y' + 4y = 0.$$

For which values of γ the system is critically damped?

$r^2 + \gamma r + 4 = 0$

$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4 \cdot 4}}{2}$

Critically damped when $\gamma^2 - 16 = 0 \Rightarrow \boxed{\gamma = 4}$

10. *(20 points)* Compute the following.
Indicate the identity that you are using.

$$\mathcal{L}\Big\{u_{2\pi}(t)\underbrace{(t-2\pi)e^{(t-\pi)}\sin(t)}_{}\Big\}(s) \overset{13}{=}$$

$$\overset{13}{=} e^{-2\pi s}\mathcal{L}\left\{t\, e^{t+\pi}\sin(t+2\pi)\right\}(s) = e^{-2\pi s}\mathcal{L}\left\{t\, e^{t+\pi}\sin(t)\right\}(s) \overset{19}{=}$$

$$\overset{19\ \boxed{n=1}}{=} -e^{-2\pi s}\frac{d}{ds}\mathcal{L}\left\{e^{\pi}e^{t}\sin t\right\}(s) \overset{9\ \boxed{a=1}\ \boxed{b=1}}{=}$$

$$\overset{9}{=} -e^{-2\pi s+\pi}\frac{d}{ds}\left(\frac{1}{(s-1)^2+1^2}\right)$$

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(13) $\mathcal{L}\{u_c(t)\, f(t)\}(s) = e^{-cs}\mathcal{L}\{f(t+c)\}(s).$

(19) $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s)$

11. *(20 points)* Compute the following.
Indicate the identity that you are using.

$$\mathcal{L}^{-1}\left\{\frac{e^{-2(s-1)}}{(s-1)^3-(s-1)}\right\}(t) \overset{(14)\ \boxed{c=1}}{=\!=} e^{t}\,\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3-s}\right\}(t) \overset{(13)}{=}$$

$$\overset{(13)\ \boxed{c=2}}{=} e^{t}u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^3-s}\right\}(t-2) = e^{t}u_2(t)\,\mathcal{L}^{-1}\left\{\frac{-1}{s}+\frac{s}{s^2-1}\right\}(t-2) \overset{(1),(8)}{=}$$

$$\overset{(1),(8)\ \boxed{a=1}}{=} \boxed{e^{t}u_2(t)\left(-1+\cosh(t-2)\right)}$$

$$\frac{1}{s^3-s} = \frac{1}{s(s^2-1)} = \frac{A}{s}+\frac{Bs+C}{s^2-1} = \frac{-1}{s}+\frac{s}{s^2-1}$$

$$\left.\begin{aligned} s^2&: A+B=0\\ s&: C=0\\ 1&: -A=1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} A&=-1\\ B&=1\\ C&=0 \end{aligned}\right\}$$

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(14) $\mathcal{L}^{-1}\{F(s-c)\}(t) = e^{ct}\,\mathcal{L}^{-1}\{F(s)\}(t)$.

(13) $\mathcal{L}^{-1}\{e^{-cs}F(s)\}(t) = u_c(t)\,\mathcal{L}^{-1}\{F(s)\}(t-c)$.

12. *(20 points)* Use the method of **reduction of order** to find a second solution of the differential equation:

$$t^2y'' + 3ty' - 8y = 0, \qquad t > 0,$$

knowing that $y_1(t) = t^2$ is a solution.

Standard form of dif. eq-n: $y'' + \underbrace{\frac{3}{t}}_{p(t)} y' - \frac{8}{t^2} y = 0$

Look for a solution $y_2(t) = y_1(t)\, v(t)$.

Abel's Thm

$W(y_1, y_2) = C e^{-\int p(t)dt} = C e^{-\int \frac{3}{t} dt} = C e^{-3\ln t} = C t^{-3}$

By definition of Wronskian

$W(y_1,y_2) = W(y_1, y_1 v) = \det \begin{vmatrix} y_1 & y_1 v \\ y_1' & y_1' v + y_1 v' \end{vmatrix} = y_1^2 v'$

So, $y_1^2 v' = c t^{-3} \Rightarrow v' = \frac{c}{y_1^2} t^{-3} = \frac{c}{(t^2)^2} t^{-3} = C t^{-7} \Rightarrow$

$\Rightarrow v = \int C t^{-7} dt = t^{-6}$ (choose C apropriately)

$y_2(t) = y_1(t) v(t) = t^2 t^{-6} = t^{-4}$

$$\boxed{y_2(t) = t^{-4}}$$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28