Math 250
Summer 2009
Exam 2

NAME: ______________________________

ID No: ______________________________

This exam contains 10 questions on 9 pages (including this title page). This exam is worth a total of 100 points. The exam is broken into two parts. There are six multiple choice questions, each worth 5 points, and 4 partial credit problems. To receive full credit for a partial credit problem all work must be shown. When in doubt, fill in the details.

No notes, books or calculators may be used during the exam.

Please, Box Your Final Answer (when possible).

Problem #	Score	Maximum
1:		6 points
2:		8 points
3:		20 points
4:		8 points
5:		8 points
6:		10 points
7:		10 points
8:		10 points
9:		10 points
10:		10 points
Total:		100 points

Multiple Choice Section

1. A spring-mass system has mass 4 kg and spring constant 9 m/s^2. What is the critical value of the damping constant γ (the value for which the system goes from underdamped to overdamped state)?

(a) $\gamma = 4$ kg/s

(b) $\gamma = 6$ kg/s

(c) $\gamma = 9$ kg/s

(d) $\gamma = 12$ kg/s

$\gamma^2 - 4mk = 0$

$\gamma^2 - 4 \cdot 4 \cdot 9 = 0 \Rightarrow \gamma = 4 \cdot 3 = 12$

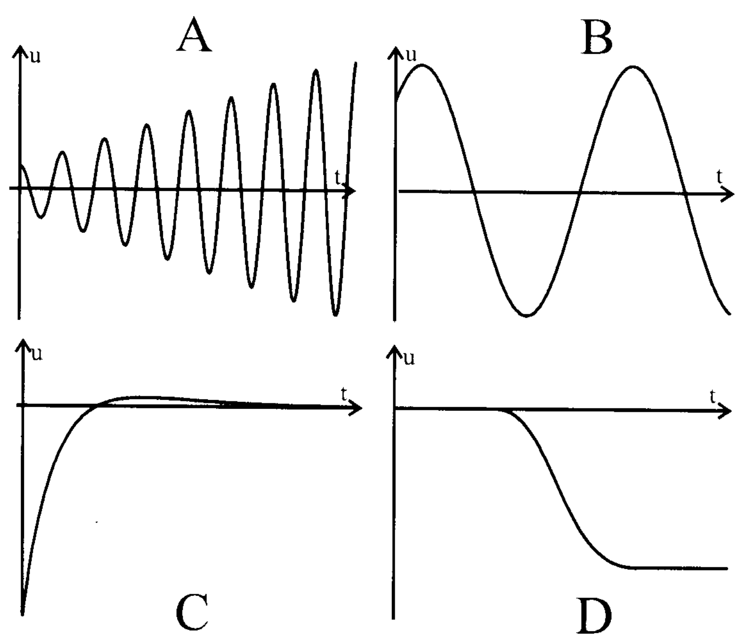
2. For each of the choices below label the corresponding graph.

(a) undumped free vibration B

(b) overdumped free vibration C

(c) resonance A

(d) something fancy (forced vibration) D



Partial Credit Section

3. Use the method of **reduction of order** to find a second solution of the differential equation:

$$t^2y'' - ty' - 3y = 0, \qquad t > 0,$$

knowing that $y_1(t) = t^3$ is a solution.
(Show all steps!)

Standard form
$y'' - \frac{1}{t}y' - \frac{3}{t^2}y = 0$ ✓

Check that y_1 is a solution: $t^2 \cdot 3 \cdot 2 \cdot t - t \cdot 3 \cdot t^2 - 3t^3 = 0$

$y_2(t) = v(t)\, y_1(t)$

$v' = \frac{1}{y_1^2} c\, e^{+\int \frac{1}{t} dt} = t^{-6} c\, t = ct^{-5}$

$v = \int ct^{-5} dt = t^{-4}$

$\boxed{y_2 = v y_1 = t^{-4} t^3 = t^{-1}}$

Check that $y_2(t) = t^{-1}$ is a solution

$(-1)(-2)t^2(t^{-3}) - t(-1)t^{-2} - 3t^{-1} =$
$= [2 + 1 - 3]\, t^{-1} = 0$ ✓

What is the **general solution** of this equation?

General solution

$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 t^3 + c_2 t^{-1}}$

4. Find the **general solution** to the differential equation:

$$y'' + 6y' + 9y = 0.$$

Characteristic eq-n: $r^2 + 6r + 9 = 0$

$(r+3)^2 = 0 \Rightarrow r_{1,2} = -3$

General solution:

$$\boxed{y(t) = c_1 e^{-3t} + c_2 t e^{-3t}}$$

5. Solve the following initial value problem

$$y''(t) - y(t) = 0, \qquad y(0) = 2, \qquad y'(0) = -1.$$

Characteristic eq-n:

$r^2 - 1 = 0 \Rightarrow r_1 = 1, \; r_2 = -1.$

General solution:

$$y(t) = c_1 e^{t} + c_2 e^{-t}$$

From initial conditions

$$\left.\begin{aligned} y(0) = 2 &= c_1 + c_2 \\ y'(0) = -1 &= c_1 - c_2 \end{aligned}\right\} +$$

$2c_1 = 1 \Rightarrow \boxed{c_1 = \frac{1}{2}}$

$\Downarrow$

$\boxed{c_2 = \frac{3}{2}}$

Solution:

$$\boxed{y(t) = \frac{1}{2} e^{t} + \frac{3}{2} e^{-t}}$$

6. Write the **form of the general solution** for the problem

$$y'' + y = \cos(t) + \cos(2t) + e^t + t^3.$$

(do not solve for undetermined coefficients!)

Solution of homogeneous eq-n $\quad y'' + y = 0$

is $\underline{C_1 \cos t + C_2 \sin t}$.

Initial guess: $Y(t) = A\cos t + B\sin t + C\cos 2t + D\sin 2t +$
$+ E e^t + F t^3 + G t^2 + H t + I$.

Repeated (with hom. sol.) terms: $A\cos t$ and $B\sin t$:

Final form: $\boxed{Y(t) = A t\cos t + B t\sin t + C\cos 2t + D\sin 2t + E e^t + F t^3 + G t^2 + H t + I}$

7. Find the solution of the initial value problem

$$y'' + 2y' - y = e^{2t}, \qquad y(0) = 1/9, \qquad y'(0) = 0.$$

Solution of homogeneous eq-n: $\quad y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0$, $\quad (r+1)^2 = 0$, $\quad r_{1,2} = -1$, $\quad y(t) = e^{-t}(c_1 + c_2 t)$.

Particular solution: $\quad Y(t) = A e^{2t}$.

Substitute:

$4Ae^{2t} + 4Ae^{2t} + Ae^{2t} = e^{2t} \quad \Big| \div e^{2t}$

$9A = 1 \quad \Rightarrow \quad \boxed{A = \frac{1}{9}}$

General solution: $\quad y(t) = e^{-t}(c_1 + c_2 t) + \frac{1}{9}e^{2t}$.

From I.C. $\quad y(0) = \frac{1}{9} = c_1 + \frac{1}{9} \quad \Rightarrow \quad \boxed{c_1 = 0}$

$y'(0) = 0 = c_2 + \frac{2}{9} \quad \Rightarrow \quad \boxed{c_2 = \frac{-2}{9}}$

Solution of I.V.P.: $\quad \boxed{y(t) = -\frac{2}{9} t e^{-t} + \frac{1}{9} e^{-2t}}$

8. A mass 10 kg stretches a spring 50 cm. It is attached to a viscous dumper that exerts a force of 15 N when the velocity of the mass is 2 m/s. If the mass is pushed up 5 cm above its equilibrium position and released, **write the differential equation with initial conditions that describes the motion of the body.**

$m = 10\ kg$

$k = \frac{mg}{L} = \frac{10\ Kg \cdot 10\ \frac{m}{s^2}}{.5\ m} = 200\ \frac{kg}{s^2}$

$\gamma = \frac{15\ N}{2\ m/s} = \frac{15}{2}\ \frac{N \cdot s}{m}$

$$\begin{cases} 10y'' + \frac{15}{2}y' + 200y = 0 \\ y(0) = -.05 \\ y'(0) = 0 \end{cases}$$

$+$

9. The motion of a body is determined by the following differential equation

$$y'' + y = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

Find the **frequency**, **amplitude** and **phase** of the oscillations.

General solution $\quad y(t) = C_1 \cos t + C_2 \sin t$

From I.C. $\quad y(0) = \boxed{1 = C_1}$

$y'(0) = \boxed{-1 = C_2}$

Solution:

$y(t) = \cos t - \sin t = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos t + \frac{-1}{\sqrt{2}}\sin t\right) =$

$= \sqrt{2}\cos\left(t - \left(\frac{-\pi}{4}\right)\right)$

Frequency	1
amplitude	$\sqrt{2}$
phase	$-\pi/4$

10. Find the **general solution** of

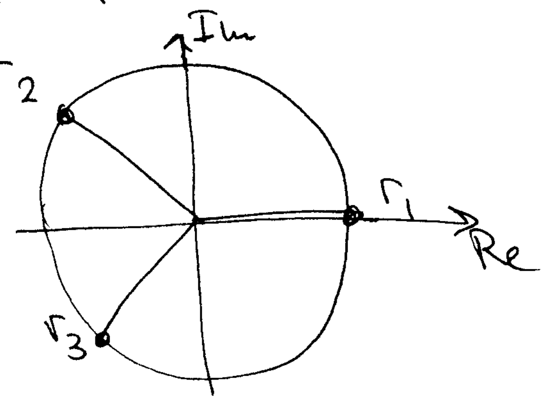
$$y^{(3)} - y = 0.$$

Characteristic eq-n: $r^3 - 1 = 0$

$r_1 = 1$

$r_2 = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$

$r_3 = \frac{-1}{2} - \frac{\sqrt{3}}{2} i$



General solution:

$$\boxed{y(t) = C_1 e^{t} + e^{-\frac{t}{2}} \left[C_2 \cos\left(\frac{\sqrt{3}}{2} t\right) + C_3 \sin\left(\frac{\sqrt{3}}{2} t\right) \right]}$$