Math 250
Summer 2009
Exam 1

NAME: ______________________

ID No: ______________________

SECTION: ______________________

This exam contains 10 questions on 9 pages (including this title page). This exam is worth a total of 100 points. The exam is broken into two parts. There are six multiple choice questions, each worth 5 points, and 4 partial credit problems. To receive full credit for a partial credit problem all work must be shown. When in doubt, fill in the details.

No notes, books or calculators may be used during the exam.

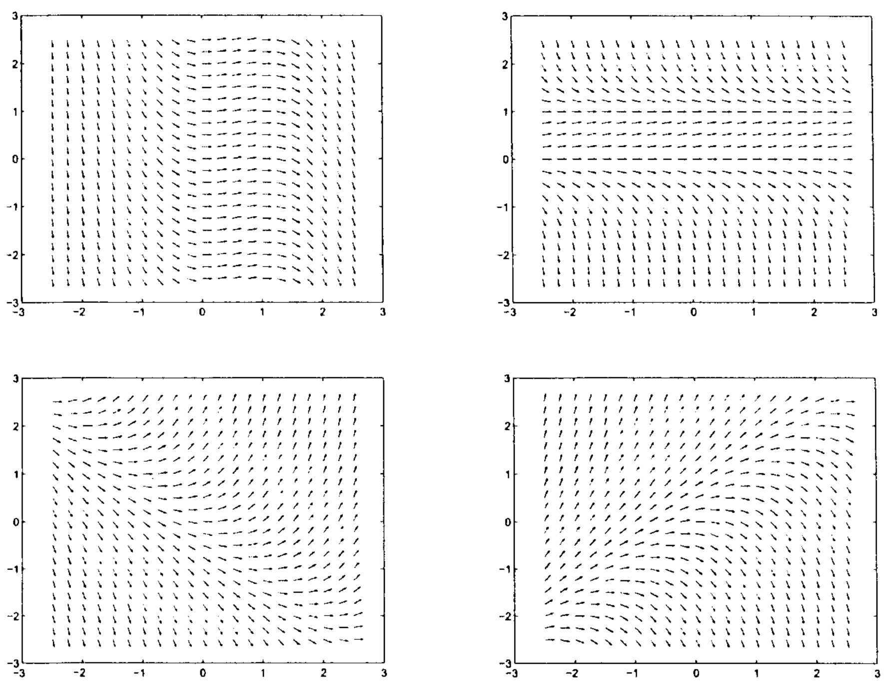
Please, Box Your Final Answer (when possible).

Problem #	Score	Maximum
1:		8 points
2:		8 points
3:		8 points
4:		8 points
5:		10 points
6:		20 points
7:		8 points
8:		8 points
9:		10 points
10:		12 points
Total:		100 points

Multiple Choice Section

1. *(8 points)* Match the differential equation with the appropriate direction field.

(A) (B)



(C) (D)

C **(a)** $y'(x) = y(x) + x$ Points $y' = 0$, $y(x) = -x$

D **(b)** $y'(x) = y(x) - x$ Points $y' = 0$, $y(x) = x$

B **(c)** $y'(x) = y(x)(1 - y(x))$ Points $y' = 0$, $y = 0$, $y = 1$

A **(d)** $y'(x) = x(1 - x)$ Points $y' = 0$, $x = 0$, $x = 1$

2. *(8 points)* Match the differential equation with the solution.

(A) $y'(x) = -2y(x)$ (B) $y'(x) = \frac{3}{x}y(x)$

(C) $y'(x) = \cos(x)$ (D) $y''(x) = 4y(x)$

(D) **(a)** $y(x) = C_1\cos(2x) + C_2\sin(2x)$

(C) **(b)** $y(x) = \sin(x) + C$

(A) **(c)** $y(x) = Ce^{-2x}$

(B) **(d)** $y(x) = x^3 + C$

3. *(8 points)* Classify the following differential equations in terms of

(i) order
(ii) linear/non-linear
(iii) ODE/PDE

	Equation	Order	L/n.L	ODE/PDE
(a)	$u_{xx}(x,y) + u_{yy}(x,y) = x^2 + y^2$	2nd	linear	PDE
(b)	$y'(x) + y^2(x) = \sin(x)$	1st	non linear	ODE
(c)	$y'(x) - xy(x) = e^x$	1st	linear	ODE
(d)	$x^2y'(x) + \sin(x)y(x) = 1$	1st	linear	ODE

Partial Credit Section

4. *(8 points)* Solve the following initial value problem:

$$xy' = y^2, \qquad y(1) = 2.$$

Separable form

$$\frac{y'}{y^2} = \frac{1}{x}$$

Integrate

$$\int \frac{y'}{y^2}\,dx = \int \frac{1}{x}\,dx$$

$$\frac{-1}{y} = \ln|x| + C$$

From I.C.

$$\frac{-1}{2} = 0 + C \Rightarrow C = \frac{-1}{2}$$

Solution

$$y = \frac{-1}{\ln|x| + C} = \frac{-1}{\ln|x| - \frac{1}{2}}$$

5. *(10 points)* Solve the following initial value problem:

$$xy' = y + x, \qquad y(0) = 1.$$ ← Ignore

Standard form for linear eq-n

$y' + \frac{-1}{x} y = 1 \quad \Big| \cdot \mu$ (use integrating factor)

$\mu y' + \frac{-1}{x} \mu y = \mu$

Find μ s.t.

$\mu' = \frac{-1}{x} \mu \Rightarrow \boxed{\mu = \frac{1}{x}}$

$(\mu y)' = \mu$

$\mu y = \int \mu \, dx + C = \int \frac{1}{x} dx + C =$

$= \ln|x| + C$

$$\boxed{y = x(\ln|x| + C)}$$

6. A tank initially contains 100 liters of fresh water. A mixture containing 10 grams of salt per liter is poured into the tank at the rate of 2 liters per minute. The well-stirred mixture is allowed to leave the tank at the same rate.

(a) *(2 points)* Introduce the variables and their meaning.

$W(t)$ – ammount of water in the tank

$S(t)$ – ammount of salt in the tank

$r_{in} = 2 \frac{lit.}{min.}$; $r_{out} = r_{in}$; $c_{in} = 10 \frac{gr}{lit}$; $c_{out} = \frac{S(t)}{W(t)}$

(b) *(4 points)* Write the differential equations, and give the initial conditions, that describe this event.

$$\begin{cases} W'(t) = r_{in} - r_{out} = 2 - 2 = 0, \quad W(0) = 100 \\ S'(t) = r_{in} c_{in} - r_{out} c_{out} = 2 \cdot 10 - 2 \frac{S(t)}{W(t)} \end{cases}$$

$$S(0) = 0$$

(c) *(8 points)* Solve the initial value problem.

$$W(t) \equiv 100 \Rightarrow$$

$$S'(t) = 20 - \frac{2}{100} S(t) \Rightarrow \quad S' + \frac{1}{50} S = 20 \quad | \cdot \mu$$

$$\mu' = \frac{1}{50}\mu \Rightarrow \boxed{\mu(t) = e^{\frac{t}{50}}}$$

$$e^{\frac{t}{50}} S \cdot = \int (e^{\frac{t}{50}} S)' \, dt = \int 20 e^{\frac{t}{50}} dt = 20 \cdot 50 \, e^{\frac{t}{50}} + C$$

$$S = 1000 + C e^{-\frac{t}{50}}$$

From I.C. $S(0) = 0 \Rightarrow C = -1000$

$$\boxed{S(t) = 1000\left(1 - e^{-\frac{t}{50}}\right)}$$

(d) *(4 points)* Find the time T when the concentration of salt in the tank is 5 grams per liter.

Concentration of salt in the tank

$$\frac{S(t)}{w(t)} = \frac{1000\left(1-e^{\frac{-t}{50}}\right)}{100} = 5 \ \frac{gr.}{lit.}$$

$$1-e^{\frac{-t}{50}} = .5 \Rightarrow e^{\frac{-t}{50}} = .5 \Rightarrow \frac{-t}{50} = \ln\left(\frac{1}{2}\right)$$

$$t = -50\ln\left(\frac{1}{2}\right) = 50\ln 2$$

$$\boxed{t = 50\ln 2}$$

(e) *(2 points)* What is the limit concentration of salt in the tank?

$$\boxed{\lim_{t\to+\infty} \frac{1000\left(1-e^{\frac{-t}{50}}\right)}{100} = 10. = C_{in}.}$$

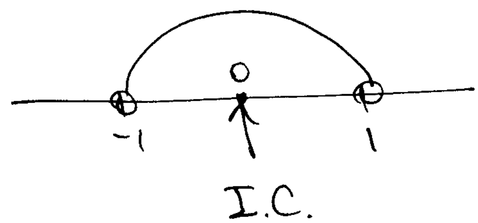
7. *(8 points)* Without solving the differential equation, find the largest interval where the initial value problem is guarantied to have a unique solution.

$$(x^2 - 1)y'(x) + e^{x^2} y(x) = x, \qquad y(0) = 3.1415.$$

Standard form

$$y' + \frac{e^{x^2}}{x^2-1} y = \frac{x}{x^2-1}$$

Problem points $x = 1$, $x = -1$



Largest interval

$(-1, 1)$.

8. *(8 points)* For which values y_0 of the initial condition the following initial value problem is guarantied to have a unique solution (locally)?

$$y'(x) = \sqrt{y}(1 + x^2), \qquad y(0) = y_0.$$

$$y'(x) = f(x,y) = \sqrt{y}\,(1+x^2)$$

- $f(x,y) = \sqrt{y}\,(1+x^2)$ is continuous everywhere $y \geq 0$.
- $f_y(x,y) = \frac{\partial}{\partial y}\left[\sqrt{y}\,(1+x^2)\right] = \frac{1}{2}\frac{1}{\sqrt{y}}(1+x^2)$

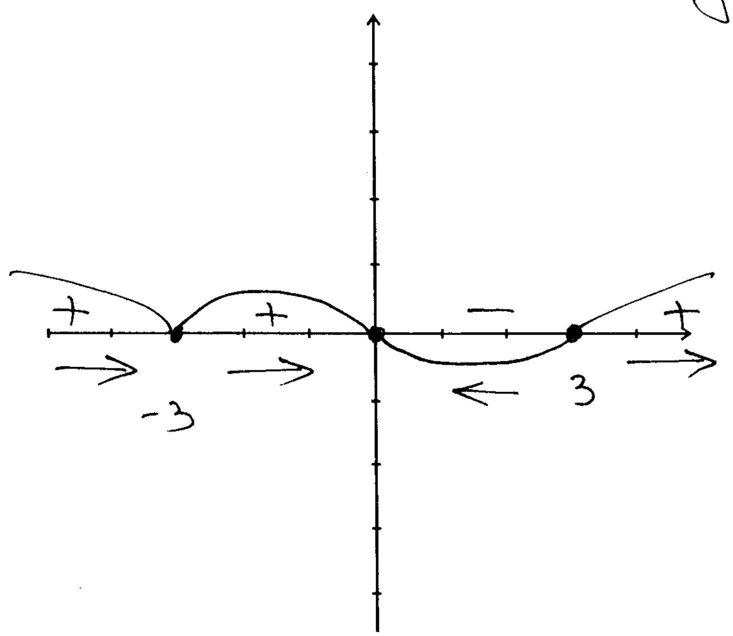
is continuous for $y > 0$.

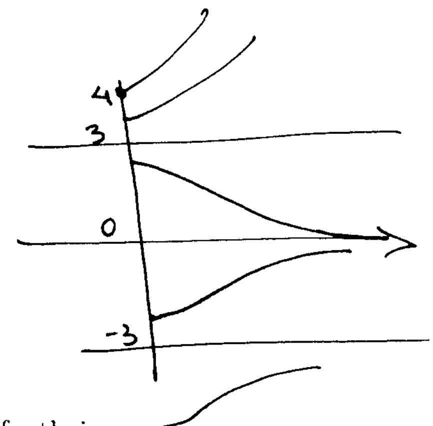
For $y_0 > 0$ there is a unique solution (locally).

9. Consider the following autonomous differential equation

$$y' = (y^2 - 9)(y + 3)y. \qquad (1)$$

(a) *(4 points)* Sketch the function $f(y) = (y^2 - 9)(y + 3)y = (y-3)(y+3)(y+3)y$





(b) *(4 points)* Identify the equilibrium solutions of (1) and classify their stability.

$y = 3$ unstable

$y = 0$ stable

$y = -3$ semistable

(c) *(2 points)* Find the limit value of the solution that satisfies the initial condition $y(-4) = 4.$

$$\lim_{t \to +\infty} y(t) = +\infty$$

10.

(a) *(4 points)* Check whether the following differential equation is exact or not. <u>Do not solve!</u>

$$\Big(2xy^3 - \sin(x)\Big) + \Big(3x^2y^3 + \cos(x)\Big)y' = 0.$$

- $\frac{\partial}{\partial y}\left(2xy^3 - \sin(x)\right) = 6xy^2$

  $\neq$ Not exact

- $\frac{\partial}{\partial x}\left(3x^2y^3 + \cos(x)\right) = 6xy^3 - \sin(x)$

(b) *(8 points)* Solve the following exact differential equation:

$$\underbrace{\Big(xy^2 - y\sin(xy) + x\Big)}_{R_x} + \underbrace{\Big(x^2y - x\sin(xy) + y\Big)}_{R_y}y' = 0.$$

$R_x = xy^2 - y\sin(xy) + x$

$R(x,y) = \frac{1}{2}x^2y^2 + \cos(xy) + \frac{1}{2}x^2 + h(y)$

$R_y = x^2y - x\sin(xy) + y$

$R(x,y) = \frac{1}{2}x^2y^2 + \cos(xy) + \frac{1}{2}y^2 + g(x)$

$R(x,y) = \frac{1}{2}x^2y^2 + \cos(x,y) + \frac{1}{2}x^2 + \frac{1}{2}y^2$

Solution (implicit)

$$\boxed{\frac{1}{2}x^2y^2 + \cos(x,y) + \frac{1}{2}x^2 + \frac{1}{2}y^2 = C}$$