NAME : ____________________

MATH 251
Summer 2002
Final Exam
Take home

ID : ____________________

INSTRUCTOR : ____________________

This exam is due at the **beginning** of class on Thursday the 1st of August, 2002. Class attendance on that day is compulsory. When you hand in this exam, please read the statement below and sign this page, returning it with your exam to your instructor. Failure to do this will result in a deduction of your final exam score.

This is a take-home final exam. You may use your notes and any book(s). However, you may not recieve any from any other person, except perhaps from myself (a clarification of a problem). By signing this you signify that you have read the above and have not breached the above conditions for this exam, if you feel you cannot fulfil these requirements please see me.

Your signature: ____________________

PLEASE DO NOT WRITE IN THE BOX BELOW.

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Total: __________

1. (a) (points) Give an example of a first order, non-linear, autonomous differential equation which has $y(t) = 0$ as an equilibrium solution.
   (b) Find a second order linear equation which has $y(t) = c_1e^{2t} + c_2e^{-3t} + t^2$ as its general solution.
   (c) Give an example of a third order, nonlinear, homogeneous *partial* differential equation.

2. (a) (points) Solve the initial value problem

$$t^2y' + 2ty = 4t^3 \qquad y(2) = 3$$

   (b) What is the largest interval on which the solution is guaranteed by the Existence and Uniqueness Theorem.

3. (points) Solve the initial value problem

$$y'' + 2y' + 10y = 10e^{-2t} \qquad y(0) = 3 \qquad y'(0) = -1$$

   You may use either then method of undetermined coefficients, or Laplace transforms to do this problem.

4. (points) A $1kg$ mass is attached to a spring with Hooke's constant of $9\frac{kg}{s^2}$ and damping constant of $6\frac{kg}{s}$. The system is set in motion from its equilibrium position with a downward velocity of $2\frac{m}{s}$. At $t = \pi$ an external force of $F(t) = \sin t$ is applied to the system, it is then disconnected at $t = 2\pi$.
   (a) Set up the initial value problem modelling this system.
   (b) Solve the initial value problem for the position function $u(t)$.
   (c) What is $u(\pi)$ and $u(3\pi)$?
   (d) Is the system overdamped, underdamped or critically damped?

5. (a) (points) Solve the initial value problem

$$\overline{X}' = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \overline{X} \qquad \overline{X}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

   (b) Classify the type and stability of the critical point (0,0).

6. (points) Consider the system

$$x' = x + y$$
$$y' = x^2 + y^2 - 8$$

   (a) Find all critical points of the system.

(b) Find the linearized matrix of the system.

(c) Classify the type and stability of each critical point.

7. (points) For the boundary value problem

$$x'' + \lambda x = 0 \qquad x(0) = 0 \qquad x'(2\pi) = 0$$

find all positive eigenvalues and their corresponding eigenfunctions.

8. (points) Let

$$f(x) = \begin{cases} 2 & 0 < x < 1 \\ 1 & 1 \le x \le 2 \end{cases}$$

(a) Sketch the odd, period 4 extension of $f(x)$ on the interval $-6 \le x \le 6$.

(b) Sketch the even, period 4 extension of $f(x)$ on the interval $-6 \le x \le 6$.

(c) Find the Fourier series of the even extension.

(d) What does the series above converge to when $x = -2$, $x = 0$ and $x = 1$?

9. (points) Solve the homogeneous heat equation:

$$\begin{aligned} 5u_{xx} &= u_t \\ u(0,t) &= u(4,t) = 0 \qquad 0 < t \\ u(x,0) &= 2\sin\left(\frac{\pi}{2}x\right) - \sin(2\pi x) - 5\sin\left(\frac{5\pi}{2}x\right) \end{aligned}$$