This exam has 13 questions for a total of 150 points. In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work. The point value for each question is in parentheses to the right of the question number.

You may not use a calculator on this exam. Please turn off and put away your cell phone.
1. (6 points) The autonomous differential equation

\[ y' = y^2 - 16 \]

has two equilibrium solutions that are

(a) asymptotically stable at \( y = -4 \), and unstable at \( y = 4 \).
(b) unstable at \( y = -4 \), and asymptotically stable at \( y = 4 \).
(c) asymptotically stable at both \( y = -4 \) and \( y = 4 \).
(d) unstable at both \( y = -4 \) and \( y = 4 \).

2. (6 points) Which equation below has the function \( y = 3e^{2t} \) as one of its solutions?

(a) \( y'' - 9y = 0 \)
(b) \( y'' + 2y = 0 \)
(c) \( y'' - 4y' + 4y = 0 \)
(d) \( y'' + 4y' + 6y = 0 \)
3. (6 points) A mass-spring system is described by the equation

\[ 4u'' + \gamma u' + 4u = 0. \]

Find all values of \( \gamma \) such that the system would be **overdamped**.

(a) \( \gamma > 8 \)
(b) \( \gamma \geq 64 \)
(c) \( 0 < \gamma < 4 \)
(d) \( 0 < \gamma < 16 \)

4. (6 points) Find the Laplace transform of \( f(t) = u_2(t)(t^2 + t - 6) \).

(a) \( \frac{e^{-2s}}{s} \left( \frac{2}{s^3} + \frac{1}{s^2} - \frac{6}{s} \right) \)
(b) \( e^{-2s} \left( \frac{2}{s^3} + \frac{1}{s^2} - \frac{6}{s} \right) \)
(c) \( \frac{e^{-2s}}{s} \left( \frac{2}{s^3} - \frac{3}{s^2} - \frac{4}{s} \right) \)
(d) \( e^{-2s} \left( \frac{2}{s^3} + \frac{5}{s^2} \right) \)
5. (6 points) The critical point at (0, 0) of the linear system

\[
\mathbf{x}' = \begin{bmatrix} 4 & -5 \\ 0 & 4 \end{bmatrix} \mathbf{x}
\]

is a(n)

(a) unstable saddle point.
(b) unstable improper (or degenerate) node.
(c) asymptotically stable proper node (star point).
(d) ( neutrally) stable center

6. (6 points) Find the Fourier sine coefficient corresponding to \( n = 2, b_2 \), of the Fourier series of the periodic function

\[
f(x) = x, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x).
\]

(a) \( b_2 = 0 \)
(b) \( b_2 = -\frac{1}{2} \)
(c) \( b_2 = -1 \)
(d) \( b_2 = 2 \)
7. (6 points) Find the steady-state solution of the heat conduction problem

\[ u_{xx} = u_t, \quad 0 < x < 10, \quad t > 0 \]

\[ u(0, t) = 100, \quad u(10, t) = 60, \]

\[ u(x, 0) = f(x). \]

(a) \( v(x) = -40x + 60 \)

(b) \( v(x) = -4x + 100 \)

(c) \( v(x) = 4x + 60 \)

(d) \( v(x) = 40x + 100 \)
8. (16 points)
   (a) (12 points) Solve the following initial value problem
   \[ ty' + 2y = 4t^2 - 2, \quad y(-1) = 5. \]

   (b) (4 points) What is the largest interval on which the solution in part (a) is guaranteed to exist uniquely?
9. (16 points) Use the method of Laplace transforms to solve the initial value problem

\[ y'' + 4y' + 8y = \delta(t - 2\pi), \quad y(0) = 1, \quad y'(0) = 0. \]
10. (16 points)

(a) (10 points) Use separation of variables to rewrite the partial differential equation below into a pair of ordinary differential equations. **DO NOT SOLVE THE EQUATIONS.**

\[ u_{tt} + 5u = 4u_{xx}. \]

(b) (4 points) Suppose the above partial differential equation has boundary condition \( u_x(0, t) = 0, u(20, t) = 0. \) Use separations of variables to determine the corresponding boundary conditions that the ordinary differential equations found in (a) must satisfy.

(c) (2 points) (Yes or no) Could the partial differential equation, \( u_{xx} - 2u_{xt} = 5u_{tt}, \) be separated into two ordinary differential equations?
11. (20 points) Find all eigenvalues and corresponding eigenfunctions of the boundary value problem

\[ X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(\pi) = 0. \]

Make sure to consider, and show your work for, all three possibilities: \( \lambda < 0, \lambda = 0, \) and \( \lambda > 0. \)
12. (20 points) Let \( f(x) = x^2, \quad 0 < x < 2. \)

(a) (4 points) Consider the odd periodic extension (of period \( T = 4 \)) of \( f(x) \). Sketch 3 periods, on the interval \(-6 < x < 6\), of this odd periodic extension.

(b) (2 points) Is the Fourier series of the periodic extension in (a) a cosine series, a sine series, or neither?

(c) (8 points) Set up, but do not integrate, the necessary integrals to find the Fourier coefficients of the periodic extension in (a).

(d) (6 points) To what value does the Fourier series converge at \( x = 0 \)? At \( x = 2 \)? At \( x = 3 \)?
13. (20 points) Suppose the temperature distribution function $u(x,t)$ of a rod that has both ends constantly kept at 0 degree is given by the heat conduction problem

$$4u_{xx} = u_t, \quad 0 < x < 6, \quad t > 0$$

$$u(0,t) = 0, \quad u(6,t) = 0,$$

$$u(x,0) = 2\sin\left(\frac{\pi x}{3}\right) + 4\sin(\pi x) - 10\sin\left(\frac{3\pi x}{2}\right).$$

(a) (18 points) Find the particular solution of the above initial-boundary value problem.

(b) (2 points) What is $\lim_{t\to\infty} u(x,t)$?