Name:____________________

MATH 251 Student Number:____________________

Final Exam Instructor:____________________

May 10, 2007 Section:____________________

This exam has **13** questions for a total of **150** points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.**

THE USE OF CALCULATORS IS NOT PERMITTED IN THIS EXAMINATION.
At the end of the examination, the booklet will be collected.

Do not write in this box.

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1. (5 points) Which of the equations below is a second order, linear, nonhomogeneous, ordinary differential equation?

   a) $(y')^2 - ty = e^t$

   b) $y'' + 2y' = \dfrac{1}{y}$

   c) $t^2y'' + 2ty' + 4y = 1$

   d) $y'' + 3y' + 2y = 0$

   e) $2y''' - 3y'' = t - 1$

2. (5 points) Find the solution of the initial value problem

$$y'' + 4y = \delta(t), \qquad y(0) = 0, \qquad y'(0) = 1.$$

   a) $y(t) = \sin 2t$

   b) $y(t) = \cos 2t + \dfrac{1}{2}\sin 2t$

   c) $y(t) = \dfrac{1}{2}\sin 2t + \dfrac{1}{2}u_1(t)\sin 2(t-1)$

   d) $y(t) = \cos t + \sin t$

   e) $y(t) = 2\sin t$

3. (5 points) Consider the initial value problem

$$t^2 y' + 2y = \frac{1}{t^2 - 16}, \qquad y(-2) = 1.$$

What is the largest interval on which a unique solution to this initial value problem is certain to exist?

a) $(-4, 4)$
b) $(-4, 0)$
c) $(0, 4)$
d) $(-\infty, 0)$
e) $(-\infty, \infty)$

4. (5 points) What is the inverse Laplace transform of $F(s) = e^{-2s}\dfrac{1}{(s-1)^2}$?

a) $\delta(t-2)te^t$
b) $u_2(t)te^t$
c) $u_2(t)(t-2)e^{2t}$
d) $u_2(t)(t-2)e^{t-2}$
e) $u_2(t)te^{t+2}$

5. (15 points) Consider systems of the form $x' = Ax$. For each pair of eigenvalues for the matrix A listed below, state the *type* and *stability* of the critical point at $(0, 0)$?

a) $r = -3, -2$

b) $r = -3, 2$

c) $r = 1 + \sqrt{5}\imath, 1 - \sqrt{5}\imath$

d) $r = 2\imath, -2\imath$

e) $r = 3, 5$

6. (10 points) Consider the following periodic functions.

$$
\begin{aligned}
a(x) &= x, & -2 \le x < 2, && a(x+4) &= a(x) \\
b(x) &= x^2, & -1 \le x \le 1, && b(x+2) &= b(x) \\
c(x) &= 1 + \cos x, & -\pi \le x < \pi, && c(x+2\pi) &= c(x) \\
d(x) &= 1 + \sin x, & -\pi \le x < \pi, && d(x+2\pi) &= d(x) \\
e(x) &= x - x^3, & -2 \le x \le 2, && e(x+4) &= e(x)
\end{aligned}
$$

a) List all the functions that can be expressed as a *Fourier sine series.*

b) List all the functions that can be expressed as a *Fourier cosine series.*

7. (10 points) Solve the initial value problem:

$$ty' + y = e^{2t}, \qquad y(1) = 4, \qquad t > 0.$$

8. (10 points) For the following boundary value problem, find all the **positive** eigenvalues and their corresponding eigenfunctions.

$$x'' + \lambda x = 0, \qquad x'(0) = 0, \qquad x(1) = 0.$$

9. (20 points) Consider the system of nonlinear equations:

$$\begin{aligned} x' &= x^2 + y^2 - 2 \\ y' &= x^2 - y^2 \end{aligned}$$

a) The system has 4 critical points. Find them.

b) One of the critical points is $(-1, -1)$. Linearize the system at that point.

c) Based on the linear system you derived in b), classify the type and stability of the critical point $(-1, -1)$.

10. (10 points) Given the partial differential equation

$$tu_{xx} - 5u_{xt} + 7u_t = 0$$

with the boundary conditions

$$u_x(0,t) = 0,\ u(L,t) = 0\ ,$$

use the technique of separation of variables to rewrite the equation and the boundary conditions in the form of two (2) ordinary differential equations with appropriate boundary and/or initial conditions.

11. (10 points) Find the *steady-state* solution, $v(x)$, of the heat conduction equation

$$\alpha^2 u_{xx} = u_t,$$

given the nonhomogeneous boundary conditions

$$u_x(0,t) + 4u(0,t) = 0, \qquad u_x(5,t) = 10.$$

12. (25 points) Consider the nonhomogeneous heat conduction problem:

$$6u_{xx} = u_t, \qquad 0 < x < 4, \qquad t > 0$$

where

$$\begin{aligned} u(0,t) &= 20 \\ u(4,t) &= 60 \\ u(x,0) &= f(x) = 10x + 20 - \sin\frac{\pi x}{2} + 15\sin 2\pi x. \end{aligned}$$

a) What is the steady-state solution $v(x)$ of this problem?

b) Find the solution, $u(x,t)$, to this problem which satisfies all the given conditions.

c) What is $\lim_{t\to\infty} u(1,t)$?

13. (20 points) Let $f(x) = -1, \qquad 0 < x < 2$

a) Expand $f(x)$ into an odd periodic function of period 4 **and** graph this periodic extension for 3 periods on the interval $[-6, 6]$.

b) Find the Fourier coefficients of the periodic function you found in a).

c) For each point $x = -1, 0$, and 3, determine the value to which the Fourier series representing $f(x)$ that you found in b) will converge.