This exam has 13 questions for a total of 150 points. In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.

THE USE OF CALCULATORS IS NOT PERMITTED IN THIS EXAMINATION. At the end of the examination, the booklet will be collected.
1. (5 points) Which of the equations below is a second order, linear, nonhomogeneous, ordinary differential equation?
   
a) \((y')^2 - ty = e^t\)
   b) \(y'' + 2y' = \frac{1}{y}\)
   c) \(t^2y'' + 2ty' + 4y = 1\)
   d) \(y'' + 3y' + 2y = 0\)
   e) \(2y''' - 3y'' = t - 1\)

2. (5 points) Find the solution of the initial value problem
   \[y'' + 4y = \delta(t), \quad y(0) = 0, \quad y'(0) = 1.\]
   
a) \(y(t) = \sin 2t\)
   b) \(y(t) = \cos 2t + \frac{1}{2} \sin 2t\)
   c) \(y(t) = \frac{1}{2} \sin 2t + \frac{1}{2} u_1(t) \sin 2(t - 1)\)
   d) \(y(t) = \cos t + \sin t\)
   e) \(y(t) = 2 \sin t\)
3. (5 points) Consider the initial value problem

\[ t^2 y' + 2y = \frac{1}{t^2 - 16}, \quad y(-2) = 1. \]

What is the largest interval on which a unique solution to this initial value problem is certain to exist?

a) \((-4, 4)\)

b) \((-4, 0)\)

c) \((0, 4)\)

d) \((-\infty, 0)\)

e) \((-\infty, \infty)\)

4. (5 points) What is the inverse Laplace transform of \( F(s) = e^{-2s} \frac{1}{(s-1)^2} \)?

a) \(\delta(t - 2)te^t\)

b) \(u_2(t)te^t\)

c) \(u_2(t)(t - 2)e^{2t}\)

d) \(u_2(t)(t - 2)e^{t-2}\)

e) \(u_2(t)te^{t+2}\)
5. (15 points) Consider systems of the form \( x' = Ax \). For each pair of eigenvalues for the matrix \( A \) listed below, state the type and stability of the critical point at \((0,0)\)?

a) \( r = -3, -2 \)

b) \( r = -3, 2 \)

c) \( r = 1 + \sqrt{5}i, 1 - \sqrt{5}i \)

d) \( r = 2i, -2i \)

e) \( r = 3, 5 \)
6. (10 points) Consider the following periodic functions.

\[ a(x) = x, \quad -2 \leq x < 2, \quad a(x + 4) = a(x) \]
\[ b(x) = x^2, \quad -1 \leq x \leq 1, \quad b(x + 2) = b(x) \]
\[ c(x) = 1 + \cos x, \quad -\pi \leq x < \pi, \quad c(x + 2\pi) = c(x) \]
\[ d(x) = 1 + \sin x, \quad -\pi \leq x < \pi, \quad d(x + 2\pi) = d(x) \]
\[ e(x) = x - x^3, \quad -2 \leq x \leq 2, \quad e(x + 4) = e(x) \]

a) List all the functions that can be expressed as a Fourier sine series.

b) List all the functions that can be expressed as a Fourier cosine series.
7. (10 points) Solve the initial value problem:

\[
Ty' + y = e^{2t}, \quad y(1) = 4, \quad t > 0.
\]
8. (10 points) For the following boundary value problem, find all the positive eigenvalues and their corresponding eigenfunctions.

\[ x'' + \lambda x = 0, \quad x'(0) = 0, \quad x(1) = 0. \]
9. (20 points) Consider the system of nonlinear equations:

\[
\begin{align*}
x' &= x^2 + y^2 - 2 \\
y' &= x^2 - y^2
\end{align*}
\]

a) The system has 4 critical points. Find them.

b) One of the critical points is \((-1, -1)\). Linearize the system at that point.

c) Based on the linear system you derived in b), classify the type and stability of the critical point \((-1, -1)\).
10. (10 points) Given the partial differential equation

\[ tu_{xx} - 5u_{xt} + 7u_t = 0 \]

with the boundary conditions

\[ u_x(0, t) = 0, \; u(L, t) = 0, \]

use the technique of separation of variables to rewrite the equation and the boundary conditions in the form of two (2) ordinary differential equations with appropriate boundary and/or initial conditions.
11. (10 points) Find the *steady-state* solution, \( v(x) \), of the heat conduction equation

\[
\alpha^2 u_{xx} = u_t,
\]

given the nonhomogeneous boundary conditions

\[
u_x(0, t) + 4u(0, t) = 0, \quad u_x(5, t) = 10.
\]
12. (25 points) Consider the nonhomogeneous heat conduction problem:

\[ 6u_{xx} = u_t, \quad 0 < x < 4, \quad t > 0 \]

where

\[
\begin{align*}
    u(0, t) &= 20 \\
    u(4, t) &= 60 \\
    u(x, 0) &= f(x) = 10x + 20 - \sin \frac{\pi x}{2} + 15 \sin 2\pi x.
\end{align*}
\]

a) What is the steady-state solution \( v(x) \) of this problem?

b) Find the solution, \( u(x, t) \), to this problem which satisfies all the given conditions.

c) What is \( \lim_{t \to \infty} u(1, t) \)?
13. (20 points) Let $f(x) = -1, \quad 0 < x < 2$

   a) Expand $f(x)$ into an odd periodic function of period 4 and graph this periodic extension for 3 periods on the interval $[-6, 6]$.

   b) Find the Fourier coefficients of the periodic function you found in a).

   c) For each point $x = -1, 0, \text{ and } 3$, determine the value to which the Fourier series representing $f(x)$ that you found in b) will converge.