

MATH 251
Final Examination
December 16, 1999

Name: _____
Student Number: _____
Instructor: _____
Section: _____

This exam has 12 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM.

Do not write in this box.

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12: _____
Total: _____

1. (12 points) Find the general solution of

$$x \frac{dy}{dx} - 4y = x^5 e^x.$$

2. (12 points) Find the general solution of the given initial value problem in **explicit form**

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = 3.$$

3. (12 points) Solve

$$y'' + 3y' + 2y = 4x^2, \quad y(0) = 7, \quad y'(0) = 0.$$

4. (12 points) Beer containing 6% alcohol per gallon is pumped into a tank that initially contains 300 gallons of beer at 3% alcohol. The rate at which the beer is pumped in is 3 gallons per minute, whereas the mixed liquid is pumped out at the same rate.
- (a) Find the number of gallons of alcohol in the tank at any time.
 - (b) What is the percentage of alcohol in the tank as $t \rightarrow \infty$?

5. (12 points) Solve

$$y'' + y = 4\delta(t - 2\pi), \quad y(0) = 1, \quad y'(0) = 0.$$

6. (12 points) Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

7. (13 points) Given the nonlinear system of differential equations

$$x' = 1 - xy, \quad y' = x^3 - y,$$

- (a) find all real critical points.
- (b) determine the stability property of each critical point.

8. (13 points) For the partial differential equation

$$yu_x + u_y = 0,$$

- (a) use **separation of variables** to change the PDE into two ordinary differential equations. Namely, let $u(x, y) = X(x)Y(y)$ and find the ordinary differential equations $X(x)$ and $Y(y)$ satisfy.
- (b) find **one** non-constant solution $u(x, y)$. (Hint: Choose a value for the separation constant and solve the ODE's you found above.)

9. (13 points) A silver bar one meter long is initially heated to a temperature of 40°C on its left half and 60°C on its right half. It is perfectly insulated at its ends and its lateral surface. For silver, the thermal diffusivity $\alpha^2 = 1.71 \text{ cm}^2/\text{sec}$.

Without solving, write down the problem that must be solved to determine the temperature $u(x, t)$ at a point x cm from the left end, at time t seconds. That is, write down the heat equation, boundary conditions and initial temperature distribution.

10. (13 points) Find the Fourier series for the function given by

$$f(x) = \begin{cases} 3 & -\pi \leq x < -\frac{\pi}{2} \\ 0 & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 3 & \frac{\pi}{2} < x \leq \pi \end{cases}$$

11. (13 points) Let the function $f(x)$ be given on $0 \leq x \leq 2$ by:

$$f(x) = \begin{cases} 0 & x = 0 \\ 2 & 0 < x < 1 \\ 2x - 2 & 1 \leq x < 2 \\ 0 & x = 2 \end{cases}$$

- (a) Let $g(x)$ be the **odd** extension of $f(x)$ with period 4. Graph $g(x)$ on the interval $-4 \leq x \leq 4$.
- (b) To what values does the Fourier series for $g(x)$ converge at $x = 0$, $x = 1$, and $x = -2$?
- (c) Let $h(x)$ be the **even** extension of $f(x)$ with period 4. Graph $h(x)$ on the interval $-4 \leq x \leq 4$.
- (d) To what values does the Fourier series for $h(x)$ converge at $x = 0$, $x = 1$, and $x = -2$?

12. (13 points) Consider an elastic string of length 10 cm whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity $u_t(x, 0) = g(x)$ where

$$g(x) = \begin{cases} 1 & 4 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

Find the displacement $u(x, t)$ which satisfies the wave equation with $\alpha = 1 \text{ cm/sec}$.

Recall: The solution of the wave equation

$$\alpha^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0,$$

satisfying the initial and boundary conditions

$$\begin{aligned} u(x, 0) &= f(x), & u_t(x, 0) &= g(x), & 0 \leq x \leq L \\ u(0, t) &= u(L, t) = 0, & t &> 0 \end{aligned}$$

has the form

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(c_n \sin \frac{n\pi\alpha t}{L} + k_n \cos \frac{n\pi\alpha t}{L} \right),$$

where

$$k_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

$$c_n = \frac{2}{n\pi\alpha} \int_0^L g(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$