MATH 251

Final Examination

December 13, 2017

FORM A

Name: ____________________

Student Number: ____________________

Section: ____________________

This exam has 17 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE AND ALL OTHER MOBILE DEVICES.

**Do not write in this box.**

| | |
|---|---|
| 1 through 12: ________ | (72) |
| 13: ________ | (15) |
| 14: ________ | (16) |
| 15: ________ | (20) |
| 16: ________ | (16) |
| 17: ________ | (11) |
| Total: ________ | |

1. (6 points) Let $y(t)$ be the solution of the initial value problem

$$y' = y^2(9 - y^2), \qquad y(2017) = -\frac{3}{2}.$$

What is $\lim_{t\to\infty} y(t)$?

(a) $-\infty$

(b) $-3$

(c) $0$

(d) $3$

2. (6 points) Consider the initial/boundary value problems below. Which of them is certain to have a unique solution for every value of $\alpha$?

**I** $y'' + 2y = 0, \quad y(\alpha) = \alpha, \quad y'(-\alpha) = -\alpha.$

**II** $y'' - 4ty = t^2, \quad y(\alpha) = \alpha, \quad y'(\alpha) = -2\alpha.$

**III** $t^2y' - ty = e^{-5t}, \quad y(\alpha) = 1 + \alpha^2.$

(a) II only.

(b) III only.

(c) Both I and II.

(d) Both II and III.

3. (6 points) Suppose $y_1(t)$ and $y_2(t)$ are two solutions of the linear differential equation

$$\sin(t)y'' + 2\cos(t)y' + ty = 0.$$

Which function below can possibly be their Wronskian, $W(y_1, y_2)(t)$?

(a) $-2\cos^2(t)$

(b) $4\sin^2(t)$

(c) $-6\sec^2(t)$

(d) $8\csc^2(t)$

4. (6 points) Consider the nonhomogeneous linear differential equation

$$y'''' - 4y'' + 3y = t.$$

Let $y_1(t) = e^{-t} + t/3$ and $y_2(t) = -2e^{\sqrt{3}t} + t/3$. Which of following statements is TRUE?

(a) $y_1$ is not a solution.

(b) $y_2$ is not a solution.

(c) Neither $y_1$ nor $y_2$ is a solution, but a linear combination of them is a solution.

(d) Both $y_1$ and $y_2$ are solutions, but some linear combinations of them are not also solutions.

5. (6 points) Consider the linear differential equation

$$y'' + 2y' + 5y = 0.$$

Which of the following statements is FALSE?

(a) $y(t) = 0$ is a solution.

(b) The functions $y_1(t) = e^{-t}\cos(4t)$ and $y_2(t) = e^{-t}\sin(4t)$ form a set of its fundamental solutions.

(c) It describes the motions of an underdamped mass-spring system.

(d) Every nonzero solution is a quasi-periodic function.

6. (6 points) Find the Laplace transform of $f(t) = u_4(t)(t-1)e^{-3t}$.

(a) $F(s) = -e^{-4s+12}\dfrac{5s+14}{(s+3)^2}$

(b) $F(s) = e^{-4s-12}\dfrac{3s+10}{(s+3)^2}$

(c) $F(s) = e^{-4s}\dfrac{1-s}{s^3(s+3)}$

(d) $F(s) = e^{-4s}\left(\dfrac{1}{(s+3)^2} + \dfrac{3}{s+3}\right)$

7. (6 points) Find the inverse Laplace transform $\mathcal{L}^{-1}\{\frac{4\pi}{3s^2+4\pi^2/3} - \frac{e^{-7s}}{2\sqrt{3}}\}$.

(a) $f(t) = 3\sin\left(\frac{2\pi t}{3}\right) - \frac{\delta(t-7)}{2\sqrt{3}}$

(b) $f(t) = 4\pi\sin\left(\frac{2\pi t}{3}\right) - \frac{\delta(t-7)}{2\sqrt{3}}$

(c) $f(t) = \left[2 - \frac{\delta(t-7)}{3}\right]\sin\left(\frac{2\pi t}{3}\right)$

(d) $f(t) = \left[\frac{4\pi}{3} - \frac{\delta(t-7)}{3}\right]\sin\left(\frac{2\pi t}{3}\right)$

8. (6 points) Suppose a certain 2×2 linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has the property that $\mathbf{x} = e^{-2t}\begin{bmatrix} t+6 \\ -t-5 \end{bmatrix}$ is a solution of the system.

Which of the following statements is TRUE?

(a) The critical point $(0,0)$ is an asymptotically stable proper node.

(b) The critical point $(0,0)$ is an unstable saddle point.

(c) $\mathbf{x} = e^{-2t}\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is another solution of the system.

(d) $\mathbf{x} = C_1 e^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-2t}\begin{bmatrix} -t-6 \\ t+5 \end{bmatrix}$ is a general solution of the system.

9. (6 points) Given that the point $(2,-2)$ is a critical point of the nonlinear system of equations

$$\begin{array}{rcl} x' &=& xy+y^2 \\ y' &=& xy+4 \end{array}.$$

This critical point $(2,-2)$ is a(n)

(a) unstable node.

(b) unstable saddle point.

(c) asymptotically stable spiral point.

(d) (neutrally) stable center.

10. (6 points) Consider the third order linear partial differential equation

$$x^2u_{xx}+2u_{ttx}=u_{xxx}.$$

Use the substitution $u(x,t)=X(x)T(t)$, where $u(x,t)$ is not the trivial solution, which of following ordinary differential equation pairs does it separate into? Please use $-\lambda$ as the separation constant.

(a) $T''-\lambda T=0, \quad X'''-\lambda(x^2X''-2X')=0.$

(b) $\lambda T''-T=0, \quad \lambda(X'''-x^2X'')+2X'=0.$

(c) $\lambda T''+T=0, \quad X'''+\lambda(x^2X''-2X')=0.$

(d) $T''+\lambda T=0, \quad X'''-x^2X''+2\lambda X'=0.$

11. (6 points) Let $u(x,t)$ be the solution of the following heat conduction initial-boundary value problem, with nonhomogeneous boundary conditions:

$$\begin{array}{ll} \pi^2 u_{xx} = u_t, & 0 < x < 3, \quad t > 0 \\ 2u_x(0,t) = 4, & u(3,t) - u_x(3,t) = 5, \\ u(x,0) = x - 2. & \end{array}$$

What is $\lim_{t\to\infty} u(0,t)$?

(a) $-2$

(b) 0

(c) 1

(d) 2

12. (6 points) Consider the Fourier series (of period $2\pi$) representing

$$f(x) = 3x \sin x - 5\sin^2 6x, \qquad -\pi < x < \pi, \qquad f(x+2\pi) = f(x).$$

Which statement below is true?

(a) The Fourier series is a cosine series.

(b) The Fourier series is a sine series.

(c) The Fourier series is neither a cosine series nor a sine series.

(d) The function does not have a Fourier series because it is not periodic.

13. (15 points) True or false:

(a) The 2-point boundary value problem, $X'' + \lambda X = 0, \quad X'(0) = 0$, and $X'(1) = 0$, does not have $\lambda = 0$ as an eigenvalue.

(b) Using the formula $u(x,t) = X(x)T(t)$, where $u(x,t)$ is not the trivial solution, the partial differential equation $2u_{tt} + e^x t^3 u_{xx} = tu_t$ cannot be separated into 2 ordinary differential equations.

(c) Using the formula $u(x,t) = X(x)T(t)$, where $u(x,t)$ is not the trivial solution, the boundary conditions $u_x(0,t) = 0$ and $u(\pi,t) = 0$ can be rewritten as $X'(0) = 0$ and $X(\pi) = 0$.

(d) Using the formula $u(x,t) = X(x)T(t)$, where $u(x,t)$ is not the trivial solution, the boundary conditions $u_x(0,t) = -1$ and $u_x(\pi,t) = 1$ cannot both be separated and simplified into expressions only in terms of $X(x)$.

(e) Every Fourier sine series (of period $T = 2L$) converges to 0 at $x = L$.

14. (16 points) Consider the two-point boundary value problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X(10) = 0.$$

(a) (12 points) Find all **positive** eigenvalues and corresponding eigenfunctions of the boundary value problem.

(b) (4 points) Is $\lambda = 0$ an eigenvalue of this problem? If yes, find its corresponding eigenfunction. If no, briefly explain why it is not an eigenvalue.

15. (20 points) Let $f(x) = 3 - x, \qquad 0 < x < 2.$

(a) (4 points) Consider the **odd** periodic extension, of period $T = 4$, of $f(x)$. Sketch 3 periods, on the interval $-6 < x < 6$, of this function.

(b) (4 points) To what value does the Fourier series of this odd periodic extension converge at $x = -3$? At $x = 8$?

(c) (4 points) Consider the **even** periodic extension, of period $T = 4$, of $f(x)$. Sketch 3 periods, on the interval $-6 < x < 6$, of this function.

(d) (3 points) Find $\dfrac{a_0}{2}$, the constant term of the Fourier series of the even periodic function described in (c).

(e) (3 points) TRUE or FALSE: For the even periodic function in part (c), the Fourier cosine coefficients $a_n$, $n \geq 1$, are given by

$$a_n = \frac{1}{2}\left(\int_{-2}^{0}(3+x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2}(3-x)\cos\left(\frac{n\pi x}{2}\right)dx\right)$$

(f) (2 points) To what value does the Fourier series of this even periodic extension converge at $x = -6$?

16. (16 points) Suppose the temperature distribution function $u(x,t)$ of a rod is given by the initial-boundary value problem

$$\begin{aligned} &2u_{xx} = u_t, \qquad 0 < x < 4, \qquad t > 0, \\ &u_x(0,t) = 0, \qquad u_x(4,t) = 0, \qquad t > 0, \\ &u(x,0) = 9 - 6\cos\left(\tfrac{5\pi x}{2}\right) + 18\cos(3\pi x), \qquad 0 < x < 4. \end{aligned}$$

(a) (2 points) What is the physical meaning of the boundary conditions?

(b) (9 points) State the general form of its solution. Then find the particular solution of the initial-boundary value problem.

(c) (2 points) What is $\lim_{t\to\infty} u(2,t)$?

(d) (3 points) Suppose the initial condition is, instead, $10 + 4\cos\left(\frac{5\pi x}{2}\right) - 18\cos(3\pi x)$. Will the limit, $\lim_{t\to\infty} u(2,t)$, be different from the temperature you found in part (c)? If yes, what is the new limit?

17. (11 points) Suppose the displacement $u(x,t)$ of a piece of flexible string is given by the initial-boundary value problem

$$\begin{aligned} &25u_{xx} = u_{tt}, \qquad 0 < x < \pi, \qquad t > 0 \\ &u(0,t) = 0, \qquad u(\pi,t) = 0, \\ &u_t(x,0) = 0, \\ &u(x,0) = \pi x - x^2. \end{aligned}$$

(a) (2 points) When $t = 0$, what is the displacement of the string at the midpoint, $x = \frac{\pi}{2}$?

(b) (4 points) In what specific form will the general solution appear?

(1) $u(x,t) = \sum_{n=1}^{\infty} C_n \sin(5nt)\sin(nx)$, (2) $u(x,t) = \sum_{n=1}^{\infty} C_n \cos(5nt)\sin(nx)$,

(3) $u(x,t) = \sum_{n=1}^{\infty} C_n \sin(nt)\sin(5nx)$, (4) $u(x,t) = \sum_{n=1}^{\infty} C_n \cos(nt)\sin(5nx)$.

Suppose the displacement $u(x,t)$ of another piece of flexible string is given by a second initial-boundary value problem

$$\begin{aligned} &25u_{xx} = u_{tt}, \qquad 0 < x < \pi, \qquad t > 0 \\ &u(0,t) = 2, \qquad u(\pi,t) = 10, \\ &u(x,0) = v(x), \\ &u_t(x,0) = 0. \end{aligned}$$

Where $v(x)$ is the string's steady-state displacement.

(c) (3 points) Use the boundary conditions to determine the steady-state displacement $v(x)$ of this string.

(d) (2 points) TRUE or FALSE: Given the scenario described by the boundary and initial conditions in this, the second, problem, the string is at a standstill and not moving at all, for all of $0 < x < \pi$ and $t > 0$.

| | $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|---|
| 1. | $1$ | $\dfrac{1}{s}$ |
| 2. | $e^{at}$ | $\dfrac{1}{s-a}$ |
| 3. | $t^n, \quad n = \text{positive integer}$ | $\dfrac{n!}{s^{n+1}}$ |
| 4. | $t^p, \quad p > -1$ | $\dfrac{\Gamma(p+1)}{s^{p+1}}$ |
| 5. | $\sin at$ | $\dfrac{a}{s^2+a^2}$ |
| 6. | $\cos at$ | $\dfrac{s}{s^2+a^2}$ |
| 7. | $\sinh at$ | $\dfrac{a}{s^2-a^2}$ |
| 8. | $\cosh at$ | $\dfrac{s}{s^2-a^2}$ |
| 9. | $e^{at}\sin bt$ | $\dfrac{b}{(s-a)^2+b^2}$ |
| 10. | $e^{at}\cos bt$ | $\dfrac{s-a}{(s-a)^2+b^2}$ |
| 11. | $t^n e^{at}, \quad n = \text{positive integer}$ | $\dfrac{n!}{(s-a)^{n+1}}$ |
| 12. | $u_c(t)$ | $\dfrac{e^{-cs}}{s}$ |
| 13. | $u_c(t)f(t-c)$ | $e^{-cs}F(s)$ |
| 14. | $e^{ct}f(t)$ | $F(s-c)$ |
| 15. | $f(ct)$ | $\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$ |
| 16. | $(f*g)(t) = \displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$ | $F(s)G(s)$ |
| 17. | $\delta(t-c)$ | $e^{-cs}$ |
| 18. | $f^{(n)}(t)$ | $s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$ |
| 19. | $(-t)^n f(t)$ | $F^{(n)}(s)$ |