MATH 251  
Final Exam  
December 20, 2007

Name:  
Student Number:  
Section:  

This exam has 15 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number.

**You may not use a calculator on this exam. Please turn off and put away your cell phone.**

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Total:  

1. (6 points) What is the form of the general solution of the equation

\[ y'' - y' - 2y = 4t^2 e^{-t} \]?

(a) \( y(t) = At^3 e^{-t} + Bt^2 e^{-t} + Cte^{-t} \)
(b) \( y(t) = t^2(A \cos t + B \sin t) \)
(c) \( y(t) = At^3 e^{-t} + Bt^2 e^{-t} + Cte^{-t} + De^{-t} + Ee^{2t} \)
(d) \( y(t) = t^2 e^{-t}(A \cos t + B \sin t) \)

2. (6 points) Find the inverse Laplace transform of the function

\[ F(s) = \frac{2(s-1)e^{-2s}}{s^2 - 2s + 2} \].

(a) \( 2u(t-1)e^t \cos t \)
(b) \( 2u(t-2)e^{t-2} \cos(t-2) \)
(c) \( 2u(t-2)e^{t-2} \sin(t-2) \)
(d) \( 2u(t-2)e^t \sin t \)
3. (6 points) Which of the following initial value problems has more than one solution?

(a) \( y' = 2y, \quad y(0) = 0 \)
(b) \( y' = y^{1/3}, \quad y(0) = 0 \)
(c) \( y'' + ty' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 0 \)
(d) \( (1 + t^2)y' + ty = 0, \quad y(0) = 0 \)

4. (6 points) Consider a pendulum with damping which is modeled by the first order system

\[
\begin{cases}
\theta' = \omega \\
\omega' = -\sin(\theta) - 3\omega
\end{cases}
\]

where \( \theta \) is the angular displacement. What is the type and stability of the critical point \((0, 0)\)?

(a) \((0, 0)\) is a center and is stable.
(b) \((0, 0)\) is an unstable saddle point.
(c) \((0, 0)\) is an asymptotically stable node.
(d) \((0, 0)\) is an asymptotically stable spiral point.
5. (6 points) Which of the following second order homogeneous linear equations has $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$ as two of its solutions?

(a) $y'' + 2y' + y = 0$
(b) $y'' - 4y' + 4y = 0$
(c) $y'' - 2y = 0$
(d) $y'' + 2ty' = 0$

6. (6 points) The explicit solution of the initial value problem

$$\frac{dy}{dt} = y^2 \cos(t) \quad , \quad y(0) = 1$$

is given by:

(a) $y = t$
(b) $y = \frac{1}{1 - \sin(t)}$
(c) $y = 2 \sqrt[3]{\frac{1}{\sin(t) + 1}}$
(d) $y = \sqrt{\frac{1}{2}(\sin^2(t) - 1)}$
7. (6 points) Which of the following functions below is a solution of the wave equation

\[ u_{tt} = 4u_{xx} \] ?

(a) \( e^{-4\pi^2t} \sin(\pi x) \)
(b) \( \sin(x - 2t) \)
(c) \( x^2 + t^2 \)
(d) \( 1 + 4 \cos(t) + x^2 \)

8. (6 points) A mass-spring system with damping is described by the initial value problem

\[ u'' + 6u' + 9u = 0, \quad u(0) = 1, \quad u'(0) = 0, \]

where \( u \) is the displacement of the mass from its equilibrium position. Then the motion of the mass is:

(a) Periodic.
(b) Oscillatory but not periodic.
(c) Critically damped and \( u(t) \) is a positive decreasing function of \( t \) for \( t > 0 \).
(d) Overdamped.
9. (6 points) Find the general solution of the linear system:

\[ x' = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} x \]

(a) \( c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^t \)

(b) \( c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} \)

(c) \( c_1 e^{6t} + c_2 e^t \)

(d) \( c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \)

10. (6 points) What is the linearization of the system

\[
\begin{align*}
x' &= x(y + 2) \\
y' &= (x - y)(y + 3)
\end{align*}
\]

around its critical point \((x, y) = (0, -3)\)?

(a) \( u' = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} u \)

(b) \( u' = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} u \)

(c) \( u' = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} u \)

(d) \( u' = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} u \)
11. (15 points) A college student borrows $5000 to buy a car. The lender charges interest at an annual rate of 10%. Assume the interest is compounded continuously and that the student makes payments continuously at a constant annual rate $k$.

(a) Let $S(t)$ denote the amount (in dollars) which the student owes at time $t$. Set up the differential equation for $S(t)$.

(b) Solve the equation in (a) for $S(t)$ using the given value of $S(0)$.

(c) Determine the payment rate $k$ that is required to pay off the loan in 5 years.
12. (20 points) Find the eigenvalues and eigenfunctions of the boundary value problem

\[ X'' + \lambda X = 0, \quad X(0) = 0, \quad X(\pi/2) = 0. \]

(Show your work in all three cases: \( \lambda = 0 \), \( \lambda < 0 \) and \( \lambda > 0 \).)
13. (15 points) (a) Apply the method of separation of variables to the partial differential equation

\[ u_{tt} + 2u_t = u_{xx} \]

and write down the resulting pair of ordinary differential equations. **Do not solve the equations.**

(b) If we consider the above partial differential equation with the boundary conditions \( u(0, t) = 0, \ u_x(4, t) = 0 \), determine the appropriate boundary conditions (if any) on the ordinary differential equations obtained in (a).
14. (20 points) Let $f$ be the periodic function with period $2\pi$ such that

$$f(x) = \begin{cases} 
2, & -\pi \leq x < 0, \\
-2, & 0 \leq x < \pi.
\end{cases}$$

(a) Find the Fourier series of the function $f$.

(b) To what value does the Fourier series in (a) converge at $x = \pi$?
15. (20 points) The temperature distribution $u(x,t)$ of a metal rod insulated at both ends is governed by the initial-boundary value problem

\[ u_t = 5u_{xx}, \quad 0 < x < 3, \quad t > 0, \]
\[ u_x(0,t) = u_x(3,t) = 0, \]
\[ u(x,0) = 10 + 4\cos(2\pi x/3) - 2\cos(4\pi x/3). \]

(a) Solve the above initial-boundary value problem for $u(x,t)$.

(b) What is the steady state temperature distribution?