MATH 251

Final Exam

December 20, 2007

Name:____________________

Student Number:____________________

Section:____________________

This exam has 15 questions for a total of 150 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

Do not write in this box.

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1. (6 points) What is the form of the general solution of the equation

$$y'' - y' - 2y = 4t^2e^{-t} \ ?$$

(a) $y(t) = At^3e^{-t} + Bt^2e^{-t} + Cte^{-t}$

(b) $y(t) = t^2(A\cos t + B\sin t)$

(c) $y(t) = At^3e^{-t} + Bt^2e^{-t} + Cte^{-t} + De^{-t} + Ee^{2t}$

(d) $y(t) = t^2e^{-t}(A\cos t + B\sin t)$

2. (6 points) Find the inverse Laplace transform of the function

$$F(s) = \frac{2(s-1)e^{-2s}}{s^2 - 2s + 2}.$$

(a) $2u(t-1)e^t\cos t$

(b) $2u(t-2)e^{t-2}\cos(t-2)$

(c) $2u(t-2)e^{t-2}\sin(t-2)$

(d) $2u(t-2)e^t\sin t$

3. (6 points) Which of the following initial value problems has more than one solution?

   (a) $y' = 2y, \quad y(0) = 0$

   (b) $y' = y^{\frac{1}{3}} \quad y(0) = 0$

   (c) $y'' + ty' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 0$

   (d) $(1 + t^2)y' + ty = 0, \quad y(0) = 0$

4. (6 points) Consider a pendulum with damping which is modeled by the first order system

$$\begin{cases} \theta' = \omega \\ \omega' = -\sin(\theta) - 3\omega \end{cases}$$

   where θ is the angular displacement. What is the type and stability of the critical point $(0, 0)$?

   (a) $(0, 0)$ is a center and is stable.

   (b) $(0, 0)$ is an unstable saddle point.

   (c) $(0, 0)$ is an asymptotically stable node.

   (d) $(0, 0)$ is an asymptotically stable spiral point.

5. (6 points) Which of the following second order homogeneous linear equations has $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$ as two of its solutions?

   (a) $y'' + 2y' + y = 0$

   (b) $y'' - 4y' + 4y = 0$

   (c) $y'' - 2y = 0$

   (d) $y'' + 2ty' = 0$

6. (6 points) The explicit solution of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 \cos(t) \quad , \quad y(0) = 1$$

   is given by:

   (a) $y = t$

   (b) $y = \frac{1}{1-\sin(t)}$

   (c) $y = 2\sqrt[3]{\frac{1}{\sin(t)+1}}$

   (d) $y = \sqrt{\frac{1}{2}(\sin^2(t) - 1)}$

7. (6 points) Which of the following functions below is a solution of the wave equation

$$u_{tt} = 4u_{xx} \ ?$$

(a) $e^{-4\pi^2 t}\sin(\pi x)$

(b) $\sin(x - 2t)$

(c) $x^2 + t^2$

(d) $1 + 4\cos(t) + x^2$

8. (6 points) A mass-spring system with damping is described by the initial value problem

$$u'' + 6u' + 9u = 0, \quad u(0) = 1,\ u'(0) = 0,$$

where u is the displacement of the mass from its equilibrium position. Then the motion of the mass is:

(a) Periodic.

(b) Oscillatory but not periodic.

(c) Critically damped and $u(t)$ is a positive decreasing function of t for $t > 0$.

(d) Overdamped.

9. (6 points) Find the general solution of the linear system:

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \boldsymbol{x}$$

(a) $c_1\binom{1}{2}e^{6t} + c_2\binom{-2}{1}e^{t}$

(b) $c_1\binom{1}{2}e^{7t} + c_2\binom{-2}{1}e^{-t}$

(c) $c_1e^{6t} + c_2e^{t}$

(d) $c_1\binom{1}{0}e^{6t} + c_2\binom{0}{1}e^{t}$

10. (6 points) What is the linearization of the system

$$\begin{cases} x' = x(y+2) \\ y' = (x-y)(y+3) \end{cases},$$

around its critical point $(x, y) = (0, -3)$?

(a) $\boldsymbol{u}' = \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix} \boldsymbol{u}$

(b) $\boldsymbol{u}' = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \boldsymbol{u}$

(c) $\boldsymbol{u}' = \begin{pmatrix} -1 & 0 \\ 6 & 3 \end{pmatrix} \boldsymbol{u}$

(d) $\boldsymbol{u}' = \begin{pmatrix} 1 & 3 \\ 6 & 9 \end{pmatrix} \boldsymbol{u}$

11. (15 points) A college student borrows \$5000 to buy a car. The lender charges interest at an annual rate of 10%. Assume the interest is compounded continuously and that the student makes payments continuously at a constant annual rate k.

(a) Let $S(t)$ denote the amount (in dollars) which the student owes at time t. Set up the differential equation for $S(t)$.

(b) Solve the equation in (a) for $S(t)$ using the given value of $S(0)$.

(c) Determine the payment rate k that is required to pay off the loan in 5 years.

12. (20 points) Find the eigenvalues and eigenfunctions of the boundary value problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X(\pi/2) = 0.$$

(Show your work in all three cases: $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$.)

13. (15 points) (a) Apply the method of separation of variables to the partial differential equation

$$u_{tt} + 2u_t = u_{xx}$$

and write down the resulting pair of ordinary differential equations. **Do not solve the equations.**

(b) If we consider the above partial differential equation with the boundary conditions $u(0,t) = 0$, $u_x(4,t) = 0$, determine the appropriate boundary conditions (if any) on the ordinary differential equations obtained in (a).

14. (20 points) Let f be the periodic function with period 2π such that

$$f(x) = \begin{cases} 2, & -\pi \le x < 0, \\ -2, & 0 \le x < \pi. \end{cases}$$

(a) Find the Fourier series of the function f.

(b) To what value does the Fourier series in (a) converge at $x = \pi$?

15. (20 points) The temperature distribution $u(x, t)$ of a metal rod insulated at both ends is governed by the initial-boundary value problem

$$\begin{aligned} &u_t = 5u_{xx}, \quad 0 < x < 3, \quad t > 0, \\ &u_x(0, t) = u_x(3, t) = 0, \\ &u(x, 0) = 10 + 4\cos(2\pi x/3) - 2\cos(4\pi x/3). \end{aligned}$$

(a) Solve the above initial-boundary value problem for $u(x, t)$.

(b) What is the steady state temperature distribution ?