Math 251

December 14, 2005 Final Exam

Name ___________________________

Section _____

There are 10 questions on this exam. Many of them have multiple parts. The point value of each question is indicated either at the beginning of each question or at the beginning of each part where there are multiple part

Where appropriate, **show your work** to receive credit; partial credit may be given.

The use of calculators, books, or notes is not permitted on this exam.

Please turn off your cell phone.

Time limit 1 hour and 50 minutes.

Question		Score
1	18pt	
2	16pt	
3	12pt	
4	14pt	
5	12pt	
6	14pt	
7	14pt	
8	16pt	
9	20pt	
10	14pt	
Total	150pt	

1. Consider the nonlinear system:

$$\begin{aligned} x' &= 2x - xy \\ y' &= -3y + xy \end{aligned}$$

a. 2pt Find all the critical points of the nonlinear system.

b. 6pt In a neighborhood of each critical point approximate the nonlinear system by a linear system.

c. 2pt Determine the name and the stability of the the critical points of each of the linear approximations.

d. 2pt Sketch a phase portrait for the original nonlinear system.

e. 2pt The linearization of a nonlinear system may have a critical point that is not guaranteed to reflect the behavior of the original nonlinear system. List the three types of critical points for which this may happen.

2. Consider the function

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 3 & \text{if} \quad 0 \le x < 1 \\ 0 & \text{if} \quad 1 \le x \end{cases}$$

a. 10pt Find the Fourier series of $f(x)$ on $[-2, 2]$. (Either use summation notation to write the answer or write the first seven terms.)

In Parts **b.** through **d.** $s_n(x)$ denotes the partial sums of the Fourier series on $[-2, 2]$ of the function $f(x)$.

b. 2pt Find $\lim_{n\to\infty} s_n(7)$

c. 2pt Find $\lim_{n\to\infty} s_n(8)$

d. 2pt Find $\lim_{n\to\infty} s_n(8.5)$

3. **a. 3pt** Which of the following already has the form of a Fourier series on the interval $[-2, 2]$. Explain!

$$f(x) = 4\sin(\frac{\pi}{3}x) \qquad\qquad g(x) = \frac{1}{4} + 2\cos(3\pi x)$$

b. 3pt We can find a sine series for the function $f(x) = x^3$ on the interval $[0, 2]$. To what value does the sine series converge at $x = 2$? Explain!

c. 3pt We can also find a cosine series for the function $f(x) = x^3$ on the interval $[0, 2]$. To what value does the cosine series converge at $x = 2$? Explain!

d. 3pt Which one of the following partial differential equations can be solved by using the technique of separation of variables?

$$u_t = u_x + 1 \qquad\qquad u_t + u_x = u$$

Explain!

4. 14pt Consider the function

$$g(x) = \begin{cases} x & \text{if} \quad 0 \le x < 1 \\ 0 & \text{if} \quad 1 \le x \le 3 \end{cases}$$

Find a cosine series for the function $g(x)$ on the interval $[0, 2]$. (Either use summation notation to write the answer or write the first four terms.)

5. 12pt Determine all POSITIVE eigenvalues λ and the corresponding eigenfunctions for the following two point boundary value problem:

$$y'' + \lambda y = 0 \qquad y'(0) = 0, \qquad y(3) = 0$$

6. Suppose a thin homogeneous rod 5 cm long is insulated along its sides and made of a material with thermal diffusivity $\alpha^2 = 0.8$ and that the left end is held at 10^o and the right end is held at 60^o.

a. 2pt What is the steady state solution to the above problem?

b. 8pt It the initial temperature of the above rod is 60^o then find the temperature $u(x,t)$ of the rod at any time $t > 0$ and at any point x inside the rod $0 < x < 5$. (If the answer involves finding a sine or cosine series then **DO NOT** find the actual values of the a_n and/or b_n which appear in the answer but indicate clearly what integrals must be evaluated to find them.)

c. 4pt For the same problem as in Part **b.**, approximately what is the temperature of the rod at $x = 3$ cm, **after a long time**.

7. **a.** **6pt** The displacement $u(x,t)$ of a string of length 5cm with ends clamped satisfies the differential equation $9u_{xx} = u_{tt}$. Suppose that the initial displacement of this string is $\sin(5\pi x)$ and the initial velocity of the string is 0. Write down a formula for the displacement $u(x,t)$ of the string at $t > 0$.
(**Hint:** $\sin(5\pi x)$ is already in the form of a Fourier series.)

a. **6pt** Consider two identical thin rods having length 6cm and which are insulated except perhaps at the their ends. Also suppose that their temperatures $u(x,t)$ satisfy the following boundary conditions.

I	**II**
$u(x,0) = 30 \quad \text{for } 0 \le x \le 6$	$u(x,0) = 30 \quad \text{for } 0 \le x \le 6$
$u(0,t) = 0 = u(6,t) \quad \text{for } t > 0$	$u_x(0,t) = 0 = u_x(6,t) \quad \text{for } t > 0$

Determine which will be warmer after a long time. Explain.

8. **a. 8pt** Find the solution of the Laplace equation on the rectangle

$$\{(x, y) \mid 0 < x < 5, \quad 0 < y < 7\}$$

which has the following values on the boundary:
$u(0, y) = 0$ if $0 < y < 7$
$u(x, 0) = 0$ if $0 < x < 5$
$u(x, 7) = 0$ if $0 < x < 5$
$u(5, y) = f(y)$ if $0 < y < 7$
(Note: Your answer requires a sine or cosine series for $f(y)$. Since no formula for $f(y)$ is given, write but do not try to evaluate the formula for a_n or b_n.)

b. 8pt Consider the function $F(\theta) = 11 + 10\cos(9\theta) + 8\sin(7\theta)$ defined on $[-\pi, \pi]$ Find the solution to the Dirichlet problem for the unit disk with the values on the boundary given by $F(\theta)$. That is, find a function in polar coordinates $u(r, \theta)$, which is a solution of Laplace's equation when $r < 1$ and which satisfies the following when $r = 1$:

$$u(1, \theta) = F(\theta) \qquad \text{for} \qquad -\pi < \theta < \pi$$

(Hint: $F(\theta)$ already has the form of a Fourier series on $[-\pi, \pi]$.

9. **a. 10 pt** Without using Laplace transforms, solve the following IVP:

$$ty' = 2y + t^3 \,, \qquad y'(1) = 3$$

b. 10pt Assume that $f(t)$ is a piecewise continuous function with Laplace transform $\mathcal{L}\{f(t)\} = F(s)$. Derive the following formula for the Laplace transform of $e^{at}f(t)$:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

10. 14pt Without using Laplace transforms, solve the following IVP:

$$y'' + 2y' - 3y = t\ , \qquad y(0) = \frac{7}{9}, \quad y'(0) = \frac{-1}{3}$$