Name: ____________
Student Number: ____________
Instructor: ____________
Section: ____________

MATH 251
Final exam
Dec. 14, 2001

There are **11** partial credit questions. Please show all of your work, as an unsupported answer will not earn any credit. A table of Laplace transforms is included at the end of the exam.

THE USE OF CALCULATORS IS NOT PERMITTED.

1. (10 pts.) Find the solution of $(1+t)y' = y$. You may leave your answer in implicit form.

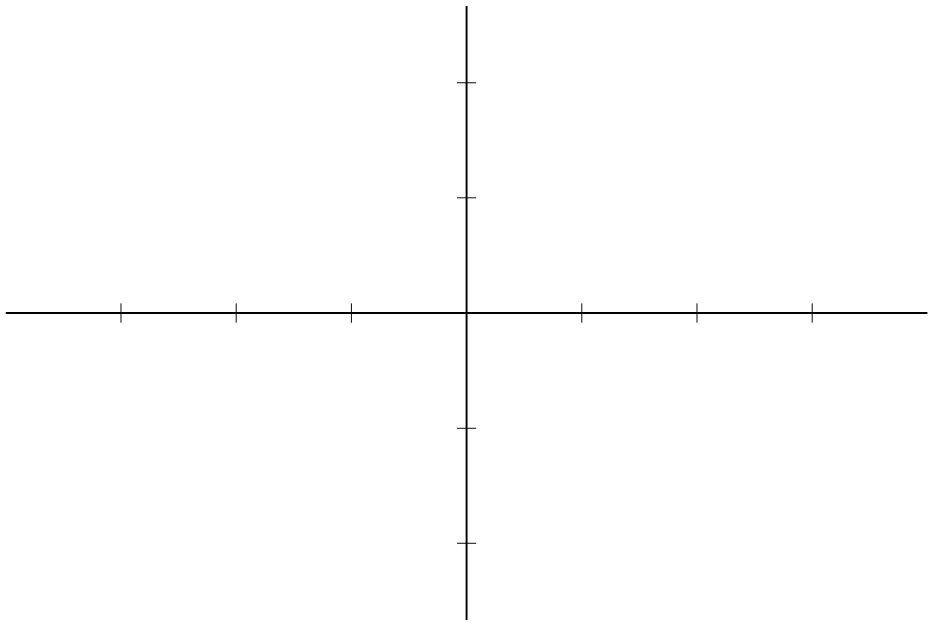
2. (15 pts.) Find the Fourier series for the function defined by

$$f(x) = \begin{cases} 0 & -4\pi \le x < -\pi \\ 3 & -\pi \le x < \pi \\ 0 & \pi \le x < 4\pi \end{cases} \quad , \qquad f(x+8\pi) = f(x).$$

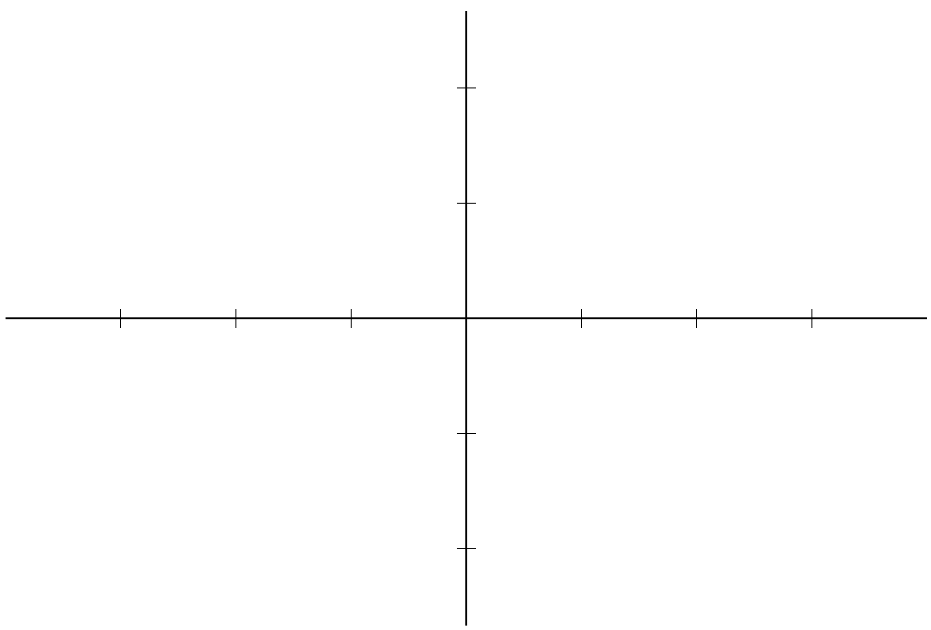
3. (10 pts.) Consider the following function, defined for $0 \leq x \leq 3$:

$$f(x) = \begin{cases} -1 & 0 \leq x \leq 2 \\ 2 & 2 < x \leq 3 \end{cases}$$

(a) Graph the even extension of $f(x)$ on the interval $-2 \leq x \leq 2$.



(b) Graph the odd extension of $f(x)$ on the interval $-2 \leq x \leq 2$.



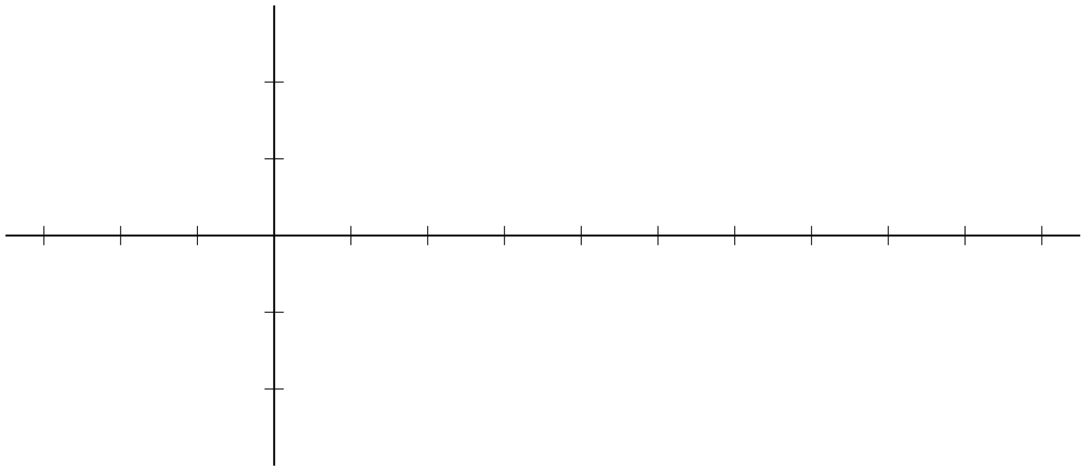
4. (15 pts.) Find the solution $y(x)$ to the initial value problem

$$(x - y + 2)dx + (y - x)dy = 0, \qquad y(0) = -1.$$

Write your answer in explicit form.

5. (10 pts.) Let $f(x) = x - 1$ on the interval $0 \leq x < 3$, and extend $f(x)$ as a function of period three by setting $f(x+3) = f(x)$.

   (a) On the supplied graph, sketch $f(x)$.



   (b) What does the Fourier series of $f(x)$ converge to at

   - $x = 0$?
   - $x = -2$?
   - $x = 3$?

6. (15 pts.) A 3 kg mass is suspended from a spring, which stretches the spring 5 m from its natural length. At $t = 0$, the system is at rest at its equilibrium position, then receives an external force of $6\cos(\omega t)$ Netwons. Assume ω is positive and $g = 10\,\text{m/s}^2$.

   (a) Find the spring constant k of the system.

   (b) Set up an *initial value problem* that describes the situation. Be sure to explain any variables that appear in your equation.

   (c) For which value of ω will the system have resonance?

7. (15 pts.) Find the general solution of

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

8. (10 pts.) Find the solution $u(x,t)$ of the heat equation

$$\begin{cases} 4u_{xx} = u_t \\ u_x(0,t) = u_x(2,t) = 0 \\ u(x,0) = 3\sin(2\pi x) - 8\sin\left(\frac{7\pi x}{2}\right), \quad 0 \leq x \leq 2 \end{cases}$$

Find $u(x,t)$. Recall that the general solution in this case is given by:

$$\sum_{n=1}^{\infty} c_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right).$$

9. (20 pts.) Consider the autonomous system of ODE's given by:

$$\begin{cases} x' = x(y-2) \\ y' = y(4-2x) \end{cases}$$

(a) Find all fixed points of the system.

(b) Find the linearization at each fixed point.

(c) Based *only* on your work in parts (a) and (b), what can you say about the type and stability of the fixed points of the nonlinear system?

10. (15 pts.) Let $y(t)$ be the solution of the initial value problem:

$$y'' - 2y' - y = tu_3(t), \qquad y(0) = 1, \quad y'(0) = -2.$$

Let $F(s) = \mathfrak{L}\{y(t)\}$ be the Laplace transform of $y(t)$. Find $F(s)$. **It is not necessary to find** $y(t)$.

11. (15 pts.) Consider the 2-point boundary value problem:

$$X'' + \lambda\pi^2 X = 0, \qquad X(0) = 0, \quad X'(1) = 0.$$

Is $\lambda = 1/4$ an eigenvalue? If not, justify your answer. If so, find a corresponding eigenfunction.