Name: ____________________

MATH 251 — Student Number: ____________________

Final Exam — Instructor: ____________________

May 7, 2004 — Section: ____________________

This exam has 12 questions for a total of 150 points.

In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.

THE USE OF CALCULATORS IS NOT PERMITTED IN THIS EXAMINATION.

The last page of this examination contains a table of Laplace transforms for your use.

Do not write in this box.

1: ________

2: ________

3: ________

4: ________

5: ________

6: ________

7: ________

8: ________

9: ________

10: ________

11: ________

12: ________

Total: ________

1. (12 points) Circle the correct answer to each of the following statements:

   (a) (2 points) True or False: The autonomous equation $y' = y^3 + y$ has exactly one real equilibrium solution.

   (b) (2 points) True or False: For nonzero constants a, b, and c, the initial value problem $ay'' + by' + cy = 0, y(0) = 1, y'(0) = 1$, always has a unique solution.

   (c) (2 points) True or False: For nonzero constants a, b, and c, the boundary value problem $ay'' + by' + cy = 0, y(0) = 1, y(5) = 1$, always has a unique solution.

   (d) (2 points) True or False: If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then $\mathcal{L}\{f(t) + g(t)\} = F(s) + G(s)$.

   (e) (2 points) True or False: If a matrix A has a repeated real eigenvalue r, then the system $x' = Ax$ has the general solution $x(t) = c_1\xi e^{rt} + c_2\xi te^{rt}$ where ξ is an eigenvector of r.

   (f) (2 points) True or False: An odd periodic function can be represented by a Fourier sine series.

2. (12 points)

   (a) (4 points) What is the integrating factor $\mu(t)$ used to solve the first order linear equation $2t^2y' - 6ty = e^{-t}$?

   (b) (4 points) If $2xy^3 + 3y\cos(xy) + (cx^2y^2 + 3x\cos(xy))y' = 0$ is an exact equation, then what is the value of c?

   (c) (4 points) What is the form of the particular solution, when the method of undetermined coefficients is used to solve $y'' - 4y = e^{-2t} + 2t^2 - 1$? Do NOT solve for the coefficients!

3. (9 points) Consider the mass-spring system described by the equation

$$u'' + \gamma u' + 9u = 0$$

.

(a) (3 points) In the absence of damping (i.e. $\gamma = 0$), what is the system's natural <u>period</u>?

(b) (3 points) If $\gamma = 4$, what is the system's quasi-frequency?

(c) (3 points) If $\gamma = 6$, is the system underdamped, critically damped, or overdamped?

4. (15 points) Solve the initial value problem

$$y'' - 6y' + 9y = \delta(t-1) - 2\delta(t-3)\,, \qquad y(0) = 0, \quad y'(0) = 0.$$

5. (8 points) Transform $2y'' - 2y' + 5y = 0$ into a system of two first order linear equations. Do NOT solve the system.

6. (18 points)

(a) (9 points) Find the general solution of $x' = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} x$.

(b) (3 points) Classify the type and stability of the critical point at $(0, 0)$.

(c) (3 points) If $x(0) = \begin{bmatrix} -3 \\ \alpha \end{bmatrix}$ and $\lim_{t\to\infty} |x(t)| = 0$, then what is the value of α?

(d) (3 points) Make a simple sketch of this system's phase portrait.

7. (15 points)

(a) (12 points) Solve the initial value problem

$$x' = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} x; \qquad x(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

.

(b) (3 points) Classify the type and stability of the critical point at $(0, 0)$.

8. (12 points) Consider the nonlinear system

$$x' = x(x - y)$$

$$y' = (x - 1)(y + 2)$$

(a) (6 points) Find all (there are 3) critical points of the system.

(b) (6 points) Choose any one of the critical points found in (a), linearize the system about the critical point, and classify its type and stability using the linearized system.

9. (10 points) Separate the partial differential equation

$$u_{xx} - u_{tx} + 5x^3 u_t = 0$$

into a system of two equations of one independent variable each.

10. (15 points)

(a) (10 points) Find all positive eigenvalues, and their corresponding eigenfunctions, of the homogeneous boundary value problem

$$y'' + \lambda y = 0; \quad y'(0) = 0, \quad y'(2\pi) = 0.$$

(b) (5 points) Is $\lambda = 0$ also an eigenvalue? If yes, give an example of its eigenfunction.

11. (9 points) Let

$$f(x) = \begin{cases} -2-x, & -2 < x < -1 \\ x, & -1 \le x \le 1 \\ 2-x, & 1 < x < 2. \end{cases} \qquad f(x+4) = f(x)$$

Set up, but do NOT integrate, the integral(s) to find the Fourier coefficients of this function.

12. (15 points) Find the particular solution of the homogeneous heat conduction problem:

$$5u_{xx} = u_t, \qquad 0 < x < 4, \qquad t > 0,$$

$$u(0,t) = 0, \qquad u(4,t) = 0,$$

$$u(x,0) = f(x) = 2\sin\frac{\pi x}{4} - \sin \pi x$$