NAME : ______________________

MATH 251
Fall 2003
Final Exam
December 15, 2003

ID : ______________________

INSTRUCTOR : ______________________

There are **14** questions on **13** pages. Please read each problem carefully before starting to solve it. For each multiple choice problem 4 answers are given, only one of which is correct. Mark only one choice. For partial credit questions, all work must be shown - **credit will not be given for an answer unsupported by work.**

NO CALCULATORS ARE ALLOWED.
PLEASE DO NOT WRITE IN THE BOX BELOW.

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Total: ________

1. (6 points) What is the inverse Laplace transform of

$$F(s) = \frac{4e^{-2s}}{s^2+4}.$$

   (a) $u_2(t)\sin(2t-2)$
   (b) $2u_2(t)\sin(2t-4)$
   (c) $4\sin(2t-4)$
   (d) $2u_3(t)\sin(2t-2)$

2. (6 points) Consider the following ordinary differential equation. Choose the **wrong** statement from the choices below.

$$\frac{dy}{dt} = y^3 - 6y.$$

   (a) Equilibrium solutions are $y = 0, \sqrt{6}, -\sqrt{6}$.
   (b) If $y(0) = \sqrt{7}$, then $\lim_{t\to\infty} y(t) = 0$.
   (c) If $y(0) = \sqrt{6}$, then $\lim_{t\to\infty} y(t) = 0$.
   (d) 0 is the only stable solution.

3. (6 points) Find the value a for which the given equation below is exact.

$$(xy^2 + ax^2y) + (x+y)x^2\frac{dy}{dx} = 0$$

(a) $a = 1$
(b) $a = 2$
(c) $a = 3$
(d) $a = 4$

4. (6 points) Find the correct step-function representation of the following function:

$$f(t) = \begin{cases} t & t < 1 \\ e^t + e^{-t} & 1 \le t < 2 \\ 1 & 2 \ge t \end{cases}$$

(a) $t + u_1(t)(-t + e^t + e^{-t}) + u_2(t)(1 - e^t - e^{-t})$
(b) $1 + u_1(t)(-1 + e^t + e^{-t}) + u_2(t)(t - e^t - e^{-t})$
(c) $1 + u_1(t)(e^t + e^{-t}) - u_2(t)(e^t + e^{-t})$
(d) $u_1(t)(-t + e^t + e^{-t}) + u_2(t)(1 - e^t - e^{-t})$

5. (6 points) Consider the Fourier series of the function given below.

$$f(x) = \begin{cases} x^2+1 & -2 < x < \frac{1}{2} \\ \sin(2\pi x) & \frac{1}{2} \leq x < 2 \end{cases}, \quad f(x+4) = f(x)$$

To what value does the Fourier series converges to at $x = \frac{1}{2}$?

(a) $\frac{5}{4}$

(b) $\frac{5}{8}$

(c) 0

(d) $\frac{1}{2}$

6. (6 points) Which of the following is the general solution to

$$X' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} X.$$

(a) $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5t}$

(b) $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t$

(c) $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} e^{5t}$

(d) $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5t} + c_2 \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} e^t$

7. (a) (14 points) Find the critical points for the system

$$\begin{aligned} \frac{dx}{dt} &= x^2 - y^2 \\ \frac{dy}{dt} &= 1 - x \end{aligned}$$

(b) Linearize the system at any one critical point you like. Specify the stability of the critical point. (stable/asymptotically stable/unstable).

8. (12 points) Solve the following initial value problem.

$$y'' + 2y' - 3y = t + \delta(t), \qquad y(0) = 0 \qquad y'(0) = 0$$

9. (14 points) A mass of $2kg$ stretches a spring $5m$. The system has a damping constant of $6\frac{kg}{s}$. The mass is initially at its equilibrium position and set in motion with downward velocity $5\frac{m}{s}$. You may take $g = 10\frac{m}{s^2}$.

   (a) Construct and solve the differential equation for this system.

   (b) What is the quasi-period of this system?

10. (a) (14 points) Find the critical points for the equation

$$\frac{dy}{dt} = y(9 - y^2) \qquad y(0) = y_0\,,\ -\infty < y_0 < \infty$$

(b) Classify the stability of the critical points as asymptotically stable/unstable.

11. (12 points) Rewrite the following function in terms of step functions and find its Laplace transform.

$$f(t) = \begin{cases} 1 & 0 \leqslant t < \frac{\pi}{2} \\ \sin 2t & \frac{\pi}{2} \leqslant t \end{cases}$$

12. (16 points) Find the Fourier series for the following function

$$f(x) = \pi^2 - x^2 \qquad\qquad -\pi \le x \le \pi$$

$$f(x + 2\pi) = f(x)$$

13. (16 points) Find the solution to the Heat Equation with boundary conditions given below :

$$\begin{aligned} u_t &= 36u_{xx} \qquad 0 < x < 4, \qquad t > 0 \\ u(0,t) &= 0 = u(4,t) \qquad t > 0 \\ u(x,0) &= \sin \pi x - \sin \frac{5\pi x}{4} \end{aligned}$$

14. (16 points) Solve the initial value problem

$$X' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} X \qquad\qquad X(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$