MATH 251

Examination II

July 25, 2011

FORM A

Name:__________________

Student Number:__________________

Section:__________________

This exam has 10 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE AND ALL OTHER MOBILE DEVICES.

Do not write in this box.

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Total:__________

1. (6 points) Find the general solution of the linear equation

$$y^{(5)} + 2y^{(4)} + 5y''' = 0.$$

2. (5 points) Rewrite the following third order linear equation into an equivalent system of first order linear equations.

$$y''' + 3y'' - 2y' + 4y = \sin 2t$$

3. (8 points)

   (a) (4 points) Evaluate the following definite integral $\int_0^\infty e^{(4-s)t}\cos(3t)\,dt$.

   (Hint: Use the fact that this integral represents the Laplace transform of a certain function. Avoid computing it directly.)

   (b) (4 points) Suppose $\mathcal{L}\{f(t)\} = \dfrac{2s^2}{s^4+100}$. Use properties of the Laplace transform to determine $\mathcal{L}\{e^{-\pi t}f(t)\}$.

4. (5 points) Suppose the linear system $\mathbf{x}' = \begin{bmatrix} 2-\alpha^2 & 0 \\ 0 & -2\alpha-1 \end{bmatrix}\mathbf{x}$ has an unstable proper node at $(0,0)$. Determine all possible value(s) of α.

5. (12 points) For each part below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2x2 matrix of real numbers. Based solely on the information given in each part, determine the type and stability of the system's critical point at $(0,0)$.

   (a) Eigenvalues of $\mathbf{A}$ are -1 and -6.

   (b) Eigenvalues of $\mathbf{A}$ are $3+7i$ and $3-7i$.

   (c) Eigenvalues of $\mathbf{A}$ are $9i$ and $-9i$.

   (d) Eigenvalues of $\mathbf{A}$ are $\sqrt{11}$ and π^2.

   (e) The general solution is $\mathbf{x}(t) = C_1 e^{-5t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-5t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (f) The general solution is $\mathbf{x}(t) = C_1 e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{t}\begin{bmatrix} -t \\ -\sqrt{3}+t \end{bmatrix}$.

6. (14 points) Find the inverse Laplace transform of each function given below.

(a) (7 points) $$F(s) = \frac{s^2 + 11s - 31}{(s+5)(s^2+36)}$$

(b) (7 points) $$F(s) = e^{-s}\frac{-5s+2}{s^2+4s+20}$$

7. (12 points) Rewrite the following piecewise continuous function $f(t)$ in terms of the unit-step function.

$$f(t) = \begin{cases} 2t^2 - e^{-6t}, & 0 \leq t < 4 \\ 9t + 3, & 4 \leq t \end{cases} .$$

Then find its Laplace transform.

8. (14 poins) Consider the initial value problem

$$y'' + 4y = \delta(t-\pi) + u_{10}(t), \qquad y(0) = 1, \qquad y'(0) = -4.$$

(a) (12 points) Use the Laplace transform to solve the initial value problem.

(b) (1 point) Evaluate $y(\frac{\pi}{2})$.

(c) (1 point) Evaluate $y(2\pi)$.

9. (12 points)

(a) (10 points) Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

(b) (2 points) Classify the type and stabiliy of the critical point at $(0, 0)$.

10. (12 points) Consider the nonlinear system:

$$\begin{aligned} x' &= (x+1)(y-2) &= xy - 2x + y - 2 \\ y' &= (x+y)(2x-y) &= 2x^2 + xy - y^2 \end{aligned}$$

(a) (4 points) The system has 4 critical points. One of the critical points of the system is $(-1, 1)$. Find the other 3 critical points of the system.

(b) (4 points) Linearize the system about the critical point $(-1, 1)$. Identify the coefficient matrix of the linearized system.

(c) (4 points) What are the eigenvalues of the coefficient matrix? Classify the type and stability of the critical point at $(-1, 1)$ by examining the linearized system found in (b).