This exam has 9 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown.** Credit will not be given for an answer not supported by work. The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

**Please turn off and put away your cell phone.**

**You may not use a calculator on this exam.**

Do not write in this box.
1. (5 points) Which system of first order linear equations below is equivalent to the second order linear equation 

\[ y'' + 5y' - 6y = 0 \]

(a) \[ \begin{cases} x'_1 = x_2 \\ x'_2 = -5x_1 + 6x_2 \end{cases} \]

(b) \[ \begin{cases} x'_1 = x_2 \\ x'_2 = 6x_1 - 5x_2 \end{cases} \]

(c) \[ \begin{cases} x'_1 = -x_1 \\ x'_2 = 6x_1 + 5x_2 \end{cases} \]

(d) \[ \begin{cases} x'_1 = 5x_1 - 6x_2 \\ x'_2 = x_2 \end{cases} \]

2. (5 points) Suppose \( y(t) \) is the solution of the first order linear initial value problem

\[ y' + 2y = t^3 e^{-4t}, \quad y(0) = -3. \]

What is \( Y(s) \), the Laplace transform of \( y(t) \)?

(a) \( Y(s) = \frac{6}{(s + 2)(s + 4)^4} - \frac{3}{s + 2} \)

(b) \( Y(s) = \frac{6}{(s + 2)(s + 4)^4} + \frac{3}{s + 2} \)

(c) \( Y(s) = \frac{3}{(s + 2)(s - 4)^4} + \frac{3}{s + 2} \)

(d) \( Y(s) = \frac{3}{(s + 2)(s - 4)^4} - \frac{3}{s + 2} \)
3. (13 points) Consider a mass-spring system described by the equation

\[ 2u'' + 8u' + ku = F(t), \quad k > 0. \]

Answer the following questions. Be sure to justify your answer. Full credit will not be given without supporting work.

(a) (4 points) For what value(s), or range of values, of \( k \) would the system be critically damped?

(b) (3 points) Suppose the spring was stretched 5 meters by the mass to its equilibrium position. Find the value of \( k \). You may use \( g = 10 \) as the gravitational constant.

(c) (3 points) Suppose \( F(t) = -5 \cos 2t \). Give the value(s) of \( k \), if any, such that the system would undergo resonance.

(d) (3 points) Suppose \( F(t) = 0 \) and \( k = 16 \). Find the quasi-period of the system.
4. (12 points) Find the inverse Laplace transforms of

(a) (6 points) \( \frac{3s^2 - 2s + 8}{s^3 + 4s} \)

(b) (6 points) \( e^{-3s} \frac{4s + 6}{s^2 - 6s + 25} \)
5. (12 points) Rewrite the following piecewise continuous function $f(t)$ in terms of the unit-step functions. Then find its Laplace transform $\mathcal{L}\{f(t)\}$.

$$f(t) = \begin{cases} 
1 + 2t^2, & 0 \leq t < 5 \\
e^{-4t} - t, & t \geq 5
\end{cases}$$
6. (14 points) Use the Laplace transform to solve the initial value problem

\[ y'' + 6y' + 9y = \delta(t) - 2u_5(t), \quad y(0) = 0, \quad y'(0) = -1. \]

No credit will be given if the Laplace transform is not used to solve this problem.
7. (14 points)

(a) (12 points) Solve the initial value problem

\[ x' = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}. \]

(b) (2 points) Classify the type and stability of the critical point of this system at (0, 0).
8. (12 points) In each part below, consider a certain system of two first order linear differential equations in two unknowns, $x' = Ax$.

(a) (4 points) Suppose one of the eigenvalues of the coefficient matrix $A$ is $r = 4i$, which has a corresponding eigenvector $\begin{pmatrix} 1 & -3 \\ 2 \end{pmatrix}$. Write down the system’s real-valued general solution.

(b) (2 points) Classify the type and stability of the critical point at $(0,0)$ for the system described in (a).

(c) (4 points) Suppose the coefficient matrix $A$ only has one distinct eigenvalue, $r = -7$, which has corresponding eigenvectors both $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Write down the system’s general solution.

(d) (2 points) Classify the type and stability of the critical point at $(0,0)$ for the system described in (c).
9. (13 points)

(a) (5 points) Find the critical points (there are 3) of the following nonlinear system.

\[
\begin{align*}
x' &= (x + 1)(y - 2) \\
y' &= y(x + y)
\end{align*}
\]

(b) (8 points) Linearize the following nonlinear system about its critical point \((4, 2)\) and classify its type and stability.

\[
\begin{align*}
x' &= x^2 - 4y^2 \\
y' &= xy - 2x
\end{align*}
\]