

MATH 251
Examination II
July 28, 2008

Name: _____
Student Number: _____
Section: _____

This exam has 9 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM.

Do not write in this box.

1: _____
2: _____
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7: _____
8: _____
9: _____
Total: _____

1. (5 points) Which system of first order linear equations below is equivalent to the second order linear equation

$$y'' + 5y' - 6y = 0?$$

(a) $\begin{cases} x_1' = x_2 \\ x_2' = -5x_1 + 6x_2 \end{cases}$

(b) $\begin{cases} x_1' = x_2 \\ x_2' = 6x_1 - 5x_2 \end{cases}$

(c) $\begin{cases} x_1' = -x_1 \\ x_2' = 6x_1 + 5x_2 \end{cases}$

(d) $\begin{cases} x_1' = 5x_1 - 6x_2 \\ x_2' = x_2 \end{cases}$

2. (5 points) Suppose $y(t)$ is the solution of the first order linear initial value problem

$$y' + 2y = t^3 e^{-4t}, \quad y(0) = -3.$$

What is $Y(s)$, the Laplace transform of $y(t)$?

(a) $Y(s) = \frac{6}{(s+2)(s+4)^4} - \frac{3}{s+2}$

(b) $Y(s) = \frac{6}{(s+2)(s+4)^4} + \frac{3}{s+2}$

(c) $Y(s) = \frac{3}{(s+2)(s-4)^4} + \frac{3}{s+2}$

(d) $Y(s) = \frac{3}{(s+2)(s-4)^4} - \frac{3}{s+2}$

3. (13 points) Consider a mass-spring system described by the equation

$$2u'' + 8u' + ku = F(t), \quad k > 0.$$

Answer the following questions. Be sure to justify your answer. Full credit will not be given without supporting work.

- (a) (4 points) For what value(s), or range of values, of k would the system be *critically damped*?
- (b) (3 points) Suppose the spring was stretched 5 meters by the mass to its equilibrium position. Find the value of k . You may use $g = 10$ as the gravitational constant.
- (c) (3 points) Suppose $F(t) = -5 \cos 2t$. Give the value(s) of k , if any, such that the system would undergo resonance.
- (d) (3 points) Suppose $F(t) = 0$ and $k = 16$. Find the *quasi-period* of the system.

4. (12 points) Find the inverse Laplace transforms of

(a) (6 points) $\frac{3s^2 - 2s + 8}{s^3 + 4s}$

(b) (6 points) $e^{-3s} \frac{4s + 6}{s^2 - 6s + 25}$

5. (12 points) Rewrite the following piecewise continuous function $f(t)$ in terms of the unit-step functions. Then find its Laplace transform $\mathcal{L}\{f(t)\}$.

$$f(t) = \begin{cases} 1 + 2t^2, & 0 \leq t < 5 \\ e^{-4t} - t, & t \geq 5 \end{cases}$$

6. (14 points) Use the Laplace transform to solve the initial value problem

$$y'' + 6y' + 9y = \delta(t) - 2u_5(t), \quad y(0) = 0, \quad y'(0) = -1.$$

No credit will be given if the Laplace transform is not used to solve this problem.

7. (14 points)

(a) (12 points) Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

(b) (2 points) Classify the type and stability of the critical point of this system at $(0, 0)$.

8. (12 points) In each part below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- (a) (4 points) Suppose one of the eigenvalues of the coefficient matrix \mathbf{A} is $r = 4i$, which has a corresponding eigenvector $\begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix}$. Write down the system's real-valued general solution.
- (b) (2 points) Classify the type and stability of the critical point at $(0,0)$ for the system described in (a).
- (c) (4 points) Suppose the coefficient matrix \mathbf{A} only has one distinct eigenvalue, $r = -7$, which has corresponding eigenvectors both $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Write down the system's general solution.
- (d) (2 points) Classify the type and stability of the critical point at $(0,0)$ for the system described in (c).

9. (13 points)

(a) (5 points) Find the critical points (there are 3) of the following nonlinear system.

$$\begin{cases} x' &= (x+1)(y-2) \\ y' &= y(x+y) \end{cases}$$

(b) (8 points) Linearize the following nonlinear system about its critical point $(4, 2)$ and classify its type and stability.

$$\begin{cases} x' &= x^2 - 4y^2 \\ y' &= xy - 2x \end{cases}$$