MATH 251

Examination II

April 4, 2016

FORM A

Name:________________

Student Number:________________

Section:________________

This exam has 12 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM.

Do not write in this box.

1 through 7:		(35)
8:		(10)
9:		(14)
10:		(15)
11:		(14)
12:		(12)
Total:		

1. (5 points) Let $y(t)$ be the solution of the initial value problem

$$y'' + 3y' - 5y = u_\pi(t)e^{-6t}, \qquad y(0) = -1, \qquad y'(0) = 2.$$

Find its Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$.

(a) $\dfrac{e^{-\pi s}}{(s+6)(s^2+3s-5)} + \dfrac{s+5}{s^2+3s-5}$

(b) $\dfrac{e^{-\pi(s+6)}}{(s+6)(s^2+3s-5)} - \dfrac{s+1}{s^2+3s-5}$

(c) $\dfrac{e^{-\pi(s+6)}}{(s-6)(s^2+3s-5)} + \dfrac{2s+5}{s^2+3s-5}$

(d) $\dfrac{e^{-\pi s}e^{6\pi}}{(s+6)(s^2+3s-5)} + \dfrac{s+1}{s^2+3s-5}$

2. (5 points) Consider the function

$$f(t) = \begin{cases} 0 & t < 1 \\ t\sin(t^2) & 1 \le t < 3 \\ 5t & t \ge 3 \end{cases}.$$

Which of the following expressions also describes $f(t)$?

(a) $f(t) = (u_3(t) - u_1(t))t\sin(t^2) + u_3(t)5t$

(b) $f(t) = u_1(t)t\sin(t^2) + u_3(t)5t$

(c) $f(t) = u_1(t)t\sin(t^2) + u_3(t)(t\sin(t^2) - 5t)$

(d) $f(t) = u_1(t)t\sin(t^2) + u_3(t)(5t - t\sin(t^2))$

3. (5 points) Find the Laplace transform $\mathcal{L}\{u_\pi(t)e^{2t-2\pi}(t-\pi)^2\}$.

   (a) $\dfrac{2e^{-\pi s}}{(s-2)^3}$

   (b) $\dfrac{2e^{-2\pi}e^{-\pi s}}{(s-2)^3}$

   (c) $\dfrac{e^{-\pi s}}{s(s-2)^2}$

   (d) $\dfrac{2e^{-\pi(2+s)}}{s(s-2)^3}$

4. (5 points) Which second-order linear differential equation below is equivalent to the following system

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 0 \\ \cos 4t \end{bmatrix}?$$

   (a) $y'' + 2y' - 3y = \cos 4t$

   (b) $y'' + 2y' - 3y = -\cos 4t$

   (c) $y'' - 2y' + 3y = \cos 4t$

   (d) $y'' - 2y' + 3y = -\cos 4t$

5. (5 points) Consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of real numbers. Suppose one of the eigenvalues of the coefficient matrix $\mathbf{A}$ is $r = 3 - 4i$, which has a corresponding eigenvector $\begin{bmatrix} 1 \\ 2+i \end{bmatrix}$. What is the system's real-valued general solution?

(a) $C_1 e^{3t} \begin{pmatrix} \cos 4t \\ 2\cos 4t - \sin 4t \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} \sin 4t \\ 2\sin 4t - \cos 4t \end{pmatrix}$

(b) $C_1 e^{3t} \begin{pmatrix} \sin 4t \\ 2\cos 4t + \sin 4t \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} \cos 4t \\ 2\sin 4t + \cos 4t \end{pmatrix}$

(c) $C_1 e^{3t} \begin{pmatrix} \cos 4t \\ 2\cos 4t + \sin 4t \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} \sin 4t \\ 2\sin 4t - \cos 4t \end{pmatrix}$

(d) $C_1 e^{3t} \begin{pmatrix} \sin 4t \\ 2\cos 4t - \sin 4t \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} \cos 4t \\ 2\sin 4t + \cos 4t \end{pmatrix}$

6. (5 points) Find all possible values of α such that the linear system below have an asymptotically stable improper node at $(0,0)$?

$$\mathbf{x}' = \begin{bmatrix} -3\alpha & 5 \\ -5 & -2\alpha \end{bmatrix} \mathbf{x}$$

(a) $\alpha = 10$

(b) $\alpha = -10$

(c) $\alpha = 10$ and $\alpha = -10$

(d) None of the above.

7. (5 points) Given that $(1, -1)$ is a critical point of the system

$$\begin{cases} x' = 3y + xy + 4 \\ y' = 2x - 3y - y^2 - 4 \end{cases}$$

Linearize this system about $(1, -1)$ to determine the critical point is an

(a) unstable node.

(b) asymptotically stable node.

(c) unstable saddle point.

(d) asymptotically stable spiral point.

8. (10 points) For each part below, determine whether the statement is true or false. You must justify your answers.

(a) Laplace transform has the following properties: $\mathcal{L}\{af(t)+b\,g(t)\} = a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}$, for any constants a and b; and that $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.

(b) The third derivative of the Laplace transform $\mathcal{L}\{f(t)\}$ is given by $\dfrac{d^3}{ds^3}\mathcal{L}\{f(t)\} = -\mathcal{L}\{t^3 f(t)\}$.

(c) $\mathcal{L}\{\delta(t-1)u_\pi(t)\cos(t)\} = \mathcal{L}\{\delta(t-3)u_\pi(t)e^{2t}\}$.

(d) The origin $(0,0)$ is always a critical point of a nonlinear system $\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$.

(e) Suppose $\mathbf{A}$ is a 2x2 matrix that satisfies $\mathbf{A}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$. Then $\mathbf{x}(t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{3t}$ is a solution to the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

9. (14 points) Consider the systems of linear differential equations below.

A) $\mathbf{x}' = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}$

B) $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{x}$

C) $\mathbf{x}' = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \mathbf{x}$

D) $\mathbf{x}' = \begin{bmatrix} 0 & 7 \\ -1 & 0 \end{bmatrix} \mathbf{x}$

Before you start, determine the eigenvalue(s) of each system above, and write them down in the space beside each system.
For each of parts (a) through (g) below, write down the letter corresponding to the system on this list that has the indicated property. There is only one correct answer to each part. However, a system may be re-used for more than one part.

a) Which system is (neutrally) stable?

b) Which system has a proper node at $(0, 0)$?

c) Every nonzero vector is an eigenvector of this system's coefficient matrix.

d) Which system has a saddle point at $(0, 0)$?

e) This system's coefficient matrix has only one linearly independent eigenvector.

f) Which system has all its solutions converge to $(0, 0)$ as $t \to \infty$?

g) Some nonzero solutions of this system converge to $(0, 0)$, while others diverge away to infinity, as $t \to \infty$

10. (15 points) Find the inverse Laplace transform of each function given below.

(a) (7 points) $$F(s) = \frac{9-s}{s^2(s^2+9)}$$

(b) (8 points) $$F(s) = e^{-9s}\frac{s-5}{s^2-4s+13} - \frac{\sqrt{7}}{2}$$

11. (14 points) Use the Laplace transform to solve the following initial value problem.

$$y'' + 4y' + 3y = 2\delta(t-1) + u_3(t)e^{2t-6}, \qquad y(0) = 0, \qquad y'(0) = 10.$$

12. (12 points)

(a) (8 points) Find the general solution of the system of linear equations

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{x}.$$

(b) (2 points) Find the solution satisfying $\mathbf{x}(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. Completely simplify your answer into a single vector function.

(c) (2 points) Classify the type and stability of the critical point at $(0, 0)$.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n, \quad n = \text{positive integer}$	$\dfrac{n!}{s^{n+1}}$
4.	$t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\dfrac{a}{s^2+a^2}$
6.	$\cos at$	$\dfrac{s}{s^2+a^2}$
7.	$\sinh at$	$\dfrac{a}{s^2-a^2}$
8.	$\cosh at$	$\dfrac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\dfrac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
16.	$(f * g)(t) = \displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$