MATH 251

Examination II

April 7, 2014

FORM A

Name:________________

Student Number:________________

Section:________________

This exam has 12 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM.

Do not write in this box.

1 through 6:		(30)
7:		(10)
8:		(8)
9:		(15)
10:		(15)
11:		(12)
12:		(10)
Total:		

1. (5 points) Evaluate the following definite integral

$$\int_0^\infty e^{-st}\left(t^3 + \sin(2t)\right)dt.$$

(a) $\frac{3}{s^4} + \frac{2}{s^2+4}$

(b) $\frac{6}{s^4} + \frac{2}{s^2+4}$

(c) $\frac{6}{s^4} + \frac{1}{s^2+2^2}$

(d) $\frac{3}{s^3} + \frac{1}{s^2+2^2}$

2. (5 points) Consider the function

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t < \pi \\ 1 + \sin(t), & \text{if } \pi \leq t < 3\pi \\ 0, & \text{if } 3\pi \leq t \end{cases}.$$

Which of the following expressions also describes $f(t)$?

(a) $f(t) = 1 + u_\pi(t)\sin(t) + u_{3\pi}(t)(1 + \sin(t))$

(b) $f(t) = 1 + u_\pi(t)\sin(t) - u_{3\pi}(t)(\sin(t) + 1)$

(c) $f(t) = u_\pi(t)(1 + \sin(t)) - u_{3\pi}(t)\sin(t) - 1$

(d) $f(t) = u_\pi(t)(1 + \sin(t)) - u_{3\pi}(t)(1 + \sin(t))$

3. (5 points) Find the Laplace transform $\mathcal{L}\{u_3(t)(t^2-3)\}$.

(a) $F(s) = e^{-3s}\,\dfrac{2-3s^2}{s^4}$

(b) $F(s) = e^{-3s}\,\dfrac{2}{s^3}$

(c) $F(s) = e^{-3s}\,\dfrac{6s^2+6s+2}{s^3}$

(d) $F(s) = e^{-3s}\,\dfrac{6s^2-6s+2}{s^4}$

4. (5 points) Which system of first order linear equations below is equivalent to the second order linear equation

$$y'' + 4y' = 6y - 5te^{2t}?$$

(a) $\begin{cases} x_1' &= x_2 \\ x_2' &= 6x_1 - 4x_2 - 5te^{2t} \end{cases}$

(b) $\begin{cases} x_1' &= x_2 \\ x_2' &= -6x_1 + 4x_2 + 5te^{2t} \end{cases}$

(c) $\begin{cases} x_1' &= x_2 \\ x_2' &= -4x_1 + 6x_2 + 5te^{2t} \end{cases}$

(d) $\begin{cases} x_1' &= x_2 \\ x_2' &= 4x_1 - 6x_2 - 5te^{2t} \end{cases}$

5. (5 points) Consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of real numbers. Suppose one of the eigenvalues of the coefficient matrix $\mathbf{A}$ is $r = 2 + i$, which has a corresponding eigenvector $\begin{bmatrix} 2-3i \\ 4 \end{bmatrix}$. What is the system's real-valued general solution?

(a) $C_1 e^{2t} \begin{pmatrix} 2\cos t - 3\sin t \\ 4\cos t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2\sin t + 3\cos t \\ 4\sin t \end{pmatrix}$

(b) $C_1 e^{2t} \begin{pmatrix} -3\cos + 2\sin t \\ 4\sin t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3\sin t - 2\cos t \\ -4\cos t \end{pmatrix}$

(c) $C_1 e^{2t} \begin{pmatrix} 3\cos t - 2\sin t \\ -4\sin t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} -3\cos t + 2\sin t \\ 4\cos t \end{pmatrix}$

(d) $C_1 e^{2t} \begin{pmatrix} 2\cos t + 3\sin t \\ 4\cos t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} -3\cos t + 2\sin t \\ 4\sin t \end{pmatrix}$

6. (5 points) Which statements below are true?

I Every second order linear ordinary differential equation is equivalent to a corresponding 2×2 system of first order equations.

II The origin is always a critical point of a homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

III If a 2×2 matrix has a negative eigenvalue λ, then the solution of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ must be asymptotically stable.

(a) II only.

(b) I and II.

(c) I and III.

(d) I, II, and III.

7. (10 points) For each part below, determine whether the statement is true or false. You must justify your answers.

   (a) $\mathcal{L}\{e^{-2t-1}(\sin(2t)+\cos(2t))\} = \frac{1}{e}\mathcal{L}\{e^{-2t}\cos(2t)\} + \frac{1}{e}\mathcal{L}\{e^{-2t}\sin(2t)\}$.

   (b) Suppose $\mathcal{L}\{f(t)\} = \frac{1}{1+s^2}$, then $\mathcal{L}\{tf(t)\} = \frac{2s}{(1+s^2)^2}$.

   (c) Suppose $f(t) = u_2(t)\cdot t^2 - u_{2\pi}(t)\cdot e^t\cos(t)$ then $f(3) = 4$.

   (d) $\mathcal{L}\{\delta(t-\pi)\sin(2t)\} = e^{-\pi s}\frac{2}{s^2+4}$.

   (e) Suppose $\mathcal{L}\{f(t)\} = \frac{1}{s^2}e^{-s}$, then $\mathcal{L}\{e^{6t}f(t)\} = \frac{1}{(s-6)^2}\cdot e^{-s}$.

8. (8 points) For each part below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 matrix of real numbers. Based solely on the information given in each part, determine the type and stability of the system's critical point at $(0, 0)$.

   (a) The coefficient matrix is $\begin{bmatrix} -10 & 9 \\ 0 & -10 \end{bmatrix}$.

   (b) One of the eigenvalues of $\mathbf{A}$ is $r = -6i$.

   (c) The characteristic equation of A can be written as $r^2 + 6r + 10 = 0$.

   (d) The general solution is $\mathbf{x}(t) = C_1\, e^{\sqrt[3]{5}t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + C_2\, e^{\sqrt[3]{5}t} \begin{bmatrix} 2t \\ -t+4 \end{bmatrix}$.

9. (15 points) Find the inverse Laplace transform of each function given below.

(a) (7 points)
$$F(s) = \frac{4 - s}{s(s+2)^2}$$

(b) (8 points)
$$F(s) = (e^{-s} + e^{-2s}) \cdot \frac{s - 4}{s^2 - 2s + 5}$$

10. (15 points) Use the Laplace transform to solve the following initial value problem.

$$y'' + 4y' + 3y = 2\delta(t-4) + u_6(t)e^{-2(t-6)}, \qquad y(0) = 0, \qquad y'(0) = 2.$$

11. (12 points) Consider the initial value problem.

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(15) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) (10 points) Solve the initial value problem.

(b) (2 points) Classify the type and stability of the critical point at $(0, 0)$.

12. (10 points) Consider the autonomous nonlinear system:

$$\begin{aligned} x' &= x^2 - xy \\ y' &= 4xy + y^2 - 2y \end{aligned}$$

(a) (3 points) The system has 3 critical points. One of the critical points is $(0, 2)$. Verify that $(0, 2)$ is indeed a critical point. Then find the other 2 critical points.

(b) (7 points) Linearize the system about the point $(0, 2)$. Classify the type and stability of this critical point by examining the linearized system. Be sure to clearly state the linearized system's coefficient matrix and its eigenvalues.