MATH 251

Examination II

April 4, 2011

FORM A

Name:____________________

Student Number:____________________

Section:____________________

This exam has 12 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM.

Do not write in this box.

1:	
2 thru 7:	
8:	
9:	
10:	
11:	
12:	
Total:	

1. (12 points) A mass-spring system is described by the equation

$$4u'' + \gamma u' + ku = F(t).$$

(a) (3 points) Suppose the mass originally stretched the spring $2m$ to reach its equilibrium position. What is the spring constant k? (Assume $g = 10m/s^2$ to be the gravitational constant.)

(b) (3 points) Suppose $k = 25$. For what value(s) of γ would this system be critically damped?

(c) (3 points) Suppose $\gamma = 0$ and $k = 400$. What is the natural **period** of this system?

(d) (3 points) True or false: Suppose $\gamma = 10$ and $k = 10$, then every nonzero solution of the mass-spring system will cross the equilibrium position more than once. Why or why not?

2. (5 points) Consider the fourth order linear equation

$$2y^{(4)} + 50y'' = 0.$$

What is its general solution?

(a) $y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos 5t + C_4 \sin 5t$

(b) $y(t) = C_1 + C_2\, t + C_3 \cos 5t + C_4 \sin 5t$

(c) $y(t) = C_1 \cos \sqrt{5}t + C_2 \sin \sqrt{5}t + C_3\, t \cos \sqrt{5}t + C_4\, t \sin \sqrt{5}t$

(d) $y(t) = C_1 + C_2 e^t + C_3 \cos 5t + C_4 \sin 5t$

3. (5 points) Find the Laplace transform $\mathcal{L}\{u_2(t)\,(t^2+1)\}$.

(a) $F(s) = e^{-2s}\,\dfrac{s^2+2}{s^3}$

(b) $F(s) = e^{-2s}\,\dfrac{2-s^2}{s^3}$

(c) $F(s) = e^{-2s}\,\dfrac{5s^2+4s+2}{s^3}$

(d) $F(s) = e^{-2s}\,\dfrac{5s^2-4s+2}{s^3}$

4. (5 points) Evaluate the following definite integral

$$\int_0^\infty e^{-st}\delta(t-\frac{\pi}{6})\sin(3t)\,dt.$$

(a) $e^{-\frac{\pi}{6}s}$

(b) $e^{-\frac{\pi}{6}s}\dfrac{3}{s^2+9}$

(c) $e^{\frac{\pi}{6}s}\dfrac{s}{s^2+9}$

(d) $-e^{\frac{\pi}{6}s}$

5. (5 points) Which system of first order linear equations below is equivalent to the second order linear equation

$$y''+2y'-y=2t^3?$$

(a) $\begin{cases} x_1' &= x_2 \\ x_2' &= 2x_1-x_2+2t^3 \end{cases}$

(b) $\begin{cases} x_1' &= x_2 \\ x_2' &= x_1-2x_2+2t^3 \end{cases}$

(c) $\begin{cases} x_1' &= x_2 \\ x_2' &= -2x_1+x_2+2t^3 \end{cases}$

(d) $\begin{cases} x_1' &= x_2 \\ x_2' &= -x_1+2x_2+2t^3 \end{cases}$

6. (5 points) Consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of real numbers. Suppose one of the eigenvalues of the coefficient matrix $\mathbf{A}$ is $r = 1 + 3i$, which has a corresponding eigenvector $\begin{bmatrix} 2-i \\ -5 \end{bmatrix}$. What is the system's real-valued general solution?

   (a) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} 2\cos 3t + \sin 3t \\ -5\cos 3t \end{bmatrix} + C_2 e^t \begin{bmatrix} -\cos 3t + 2\sin 3t \\ -5\sin 3t \end{bmatrix}$

   (b) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} 2\cos 3t - \sin 3t \\ -5\cos 3t \end{bmatrix} + C_2 e^t \begin{bmatrix} \cos 3t - 2\sin 3t \\ 5\sin 3t \end{bmatrix}$

   (c) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} -2\cos 3t - \sin 3t \\ 5\sin 3t \end{bmatrix} + C_2 e^t \begin{bmatrix} \cos 3t - 2\sin 3t \\ -5\cos 3t \end{bmatrix}$

   (d) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} -2\cos 3t + \sin 3t \\ 5\sin 3t \end{bmatrix} + C_2 e^t \begin{bmatrix} -\cos 3t + 2\sin 3t \\ 5\cos 3t \end{bmatrix}$

7. (5 points) Consider a certain linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of real numbers with distinct nonzero real eigenvalues. Suppose all of its solutions have a finite limit as $t \to +\infty$. Then the critical point $(0, 0)$ must be a(n)

   (a) (neutrally) stable center.

   (b) asymptotically stable spiral point.

   (c) unstable saddle point.

   (d) asymptotically stable node.

8. (9 points) For each part below, determine whether the statement is true or false. You must justify your answers.

   (a) Consider the linear system $\mathbf{x}' = \begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}\mathbf{x}$. The critical point $(0,0)$ is a (neutrally) stable center.

   (b) $\mathcal{L}\{(t+5)^2\} = \mathcal{L}\{t+5\}\mathcal{L}\{t+5\}$

   (c) Suppose $f(t) = 2 + u_3(t)\,(t-1) + u_5(t)\,t^2$, then $f(4) = 5$.

9. (14 points) Find the inverse Laplace transform of each function given below.

(a) (7 points) $$F(s) = \frac{2s^2 - 1}{s^3 - 3s^2 - 10s}$$

(b) (7 points) $$F(s) = e^{-s}\frac{s}{s^2 - 10s + 29}$$

10. (14 points) Use the Laplace transform to solve the following initial value problem.

$$y'' + 4y = 2\delta(t-5) - 8u_3(t), \qquad y(0) = 0, \qquad y'(0) = 4.$$

11. (11 points) Consider the initial value problem.

$$\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

(a) (9 points) Solve the initial value problem.

(b) (2 points) Classify the type and stability of the critical point at $(0, 0)$.

12. (10 points) Consider the nonlinear system:

$$\begin{aligned} x' &= (2x+y)(2x-y) \\ y' &= (x-2)(y+2) \end{aligned}$$

(a) (3 points) One of the critical points of the system is $(1, -2)$. There are 3 other critical points. Find those other 3 critical points of the system.

(b) (7 points) Linearize the system about the point $(1, -2)$. Classify the type and stability of the critical point at $(1, -2)$ by examining the linearized system. Be sure to clearly state the linearized system's matrix and its eigenvalues.