MATH 251

Examination II

April 7, 2010

FORM A

Name:________________

Student Number:________________

Section:________________

This exam has 10 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM.

Do not write in this box.

1 thru 4:	
5:	
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10:	
Total:	

1. (5 points) A mass-spring system is initially resting at its equilibrium position with no forces acting on it. Between times $t = 2$ and $t = 5$ seconds a constant external downward force of 10 Newtons is applied. At $t = 6$ seconds the mass is struck with a hammer in such a fashion that an upward momentum of magnitude 20 kg-m/s is introduced to the system at that time. Given that the downward direction is positive, which of the following $F(t)$ represents the combination of these two forces?

   (a) $F(t) = 10u_2(t) - 10u_5(t) - 20\delta(t-6)$

   (b) $F(t) = 10u_2(t) + 10u_5(t) - 20u_6(t)$

   (c) $F(t) = 10u_5(t) - 10u_2(t) - 20(1 - u_6(t))$

   (d) $F(t) = 10\delta(t-2) - 10\delta(t-5) - 20u_6(t)$

2. (5 points) Which system of first order linear equations below is equivalent to the third order linear equation

$$y''' + y'' - 2y' + 3y = 0?$$

   (a) $\begin{cases} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -x_1 + 2x_2 - 3x_3 \end{cases}$

   (b) $\begin{cases} x_1' &= x_1 \\ x_2' &= x_2 \\ x_3' &= 3x_1 - 2x_2 + x_3 \end{cases}$

   (c) $\begin{cases} x_1' &= x_1 \\ x_2' &= x_2 \\ x_3' &= x_1 - 2x_2 + 3x_3 \end{cases}$

   (d) $\begin{cases} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -3x_1 + 2x_2 - x_3 \end{cases}$

3. (5 points) Suppose a certain linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of real numbers with nonzero eigenvalues, has at least one nonzero solution that does not reach a limit both as $t \to \infty$, and as $t \to -\infty$. Then the critical point $(0,0)$ must be

   (a) unstable.

   (b) asymptotically stable.

   (c) (neutrally) stable, but not asymptotically stable.

   (d) having indeterminate stability.

4. (5 points) All of the following systems of linear equations have exactly one critical point, at $(0,0)$. In three of the systems, the stability of $(0,0)$ are identical. In which system does $(0,0)$ have a different stability classification?

   (a) $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}$

   (b) $\mathbf{x}' = \begin{bmatrix} 0 & -4 \\ -1 & 0 \end{bmatrix} \mathbf{x}$

   (c) $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \mathbf{x}$

   (d) $\mathbf{x}' = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \mathbf{x}$

5. (14 points) Find the inverse Laplace transform of each function given below.

   (a) (7 points) $$F(s) = \sqrt{7} + \frac{4s+10}{s^2-4s+20}$$

   (b) (7 points) $$F(s) = e^{-s}\frac{1}{s^3+s^2}$$

6. (12 points) Rewrite the following piecewise continuous function $f(t)$ in terms of the unit-step function. Then find its Laplace transform.

$$f(t) = \begin{cases} 6e^{\pi t}, & 0 \leq t < 4 \\ 2t^2 - 3t, & 4 \leq t \end{cases} .$$

7. (16 points) Use the Laplace transform to solve the following initial value problem.

$$y'' + 16y = u_7(t) - \delta(t - 13), \qquad y(0) = 0, \qquad y'(0) = 1.$$

No credit will be given if the Laplace transform is not used to solve this problem.

8. (14 points)

(a) (9 points) Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

(b) (3 points) Given that $\mathbf{x}(0) = \begin{bmatrix} 9 \\ \beta \end{bmatrix}$, and $\lim_{t \to \infty} \mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, find the value of β.

(c) (2 points) Classify the type and stability of the critical point at $(0, 0)$.

9. (12 points) In the parts below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of real numbers.

   (a) (4 points) Suppose one of the eigenvalues of the coefficient matrix $\mathbf{A}$ is $r = 7 + 3i$, which has a corresponding eigenvector $\begin{bmatrix} -2-4i \\ 1 \end{bmatrix}$. Write down the system's <u>real-valued</u> general solution.

   (b) (2 points) State the type and stability of the critical point $(0, 0)$ of the system in (a).

   (c) (4 points) Consider a different system. Suppose its coefficient matrix $\mathbf{A}$ is such that it has the following matrix-vector products $\mathbf{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Write down the system's general solution.

   (d) (2 points) State the type and stability of the critical point $(0, 0)$ of the system in (c).

10. (12 points) Consider the nonlinear system:

$$\begin{aligned} x' &= x^2 - xy \\ y' &= xy + 2y^2 - 6y \end{aligned}$$

(a) (4 points) The system has 3 critical points. One of the critical points is $(0, 3)$. Find the other 2 critical points.

(b) (8 points) Linearize the system about the point $(0, 3)$. Classify the type and stability of the critical point at $(0, 3)$ by examining the linearized system. Be sure to clearly state the linearized system's matrix and its eigenvalues.