MATH 251

Examination II

April 7, 2009

Name:________________

Student Number:________________

Section:________________

This exam has 16 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM.

Do not write in this box.

1 thru 3 :__________

4 thru 13:__________

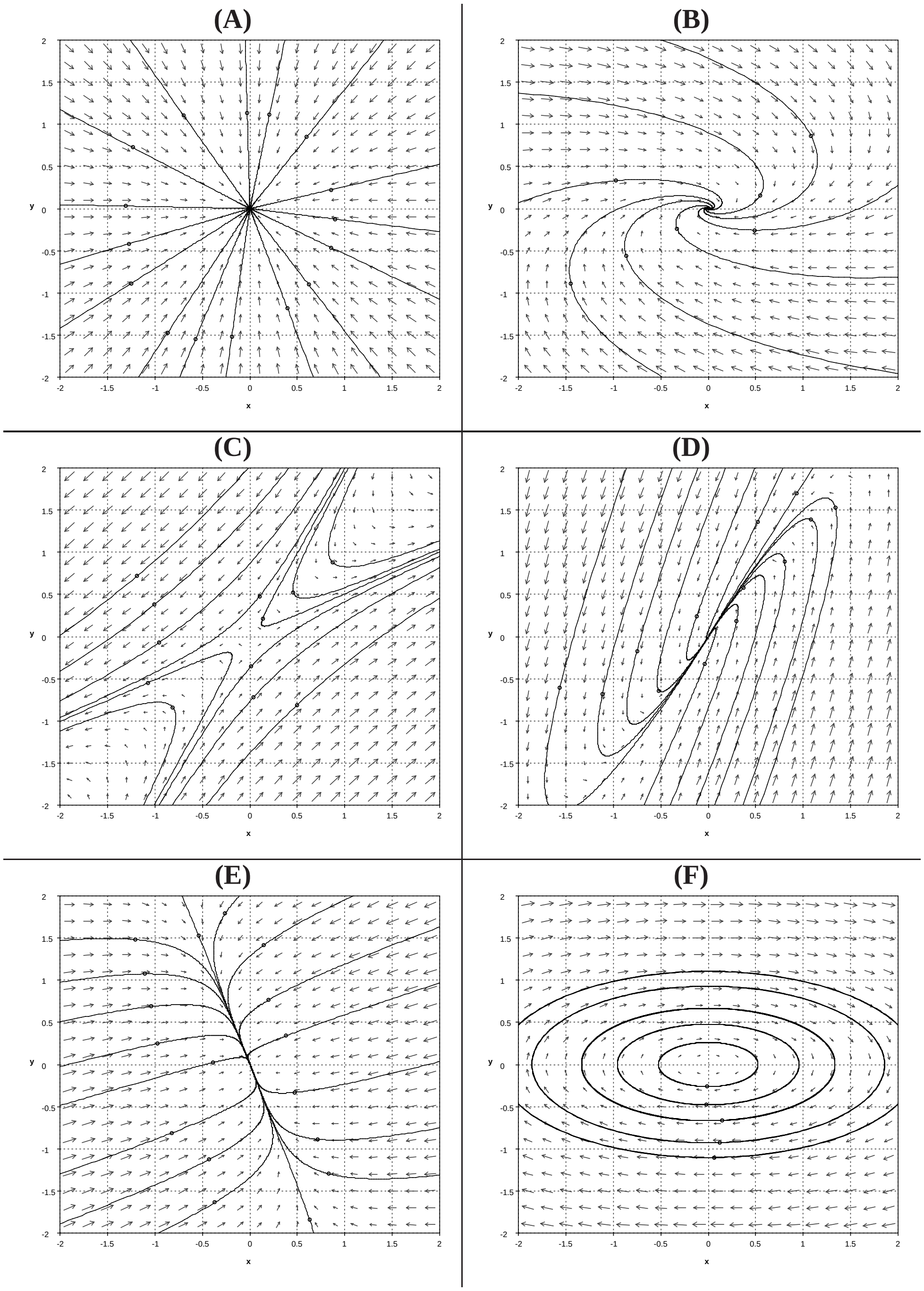
14:__________

15:__________

16:__________

Total:__________

1. (5 points) Match the sketches of phase portraits for the 2x2 linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$, lettered A through F, with the names of their critical points at the origin.



node:__________

center:__________

proper node (star point):__________

saddle point:__________

spiral point:__________

2. (5 points) Match the following general solutions of 2x2 linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with the sketches of the phase portraits given in Problem 1:

(a) $e^{-t}\left(C_1\begin{bmatrix}1\\2\end{bmatrix}+C_2\begin{bmatrix}t+1\\2t+1\end{bmatrix}\right)$ ____________

(b) $C_1e^{-t}\begin{bmatrix}1\\2\end{bmatrix}+C_2e^{-2t}\begin{bmatrix}2\\1\end{bmatrix}$ ____________

(c) $C_1e^{-t}\begin{bmatrix}1\\0\end{bmatrix}+C_2e^{-t}\begin{bmatrix}0\\1\end{bmatrix}$ ____________

(d) $C_1e^{-t}\begin{bmatrix}1\\2\end{bmatrix}+C_2e^{t}\begin{bmatrix}2\\1\end{bmatrix}$ ____________

(e) $C_1e^{t}\begin{bmatrix}2\cos t\\-\sin t\end{bmatrix}+C_2e^{t}\begin{bmatrix}2\sin t\\\cos t\end{bmatrix}$ ____________

3. (5 points) Determine the stability of the critical point at the origin for each of the 2x2 linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whose general solutions are given below. Use the letter **A** if the point is asymptotically stable, **U** if it is unstable, **S** if it is (neutrally) stable.

(a) $C_1\begin{bmatrix}2\cos t\\-\sin t\end{bmatrix}+C_2\begin{bmatrix}2\sin t\\\cos t\end{bmatrix}$ ____________

(b) $e^{-t}\left(C_1\begin{bmatrix}1\\2\end{bmatrix}+C_2\begin{bmatrix}t+1\\2t+1\end{bmatrix}\right)$ ____________

(c) $C_1e^{-t}\begin{bmatrix}1\\2\end{bmatrix}+C_2e^{t}\begin{bmatrix}2\\1\end{bmatrix}$ ____________

(d) $C_1e^{-t}\begin{bmatrix}1\\0\end{bmatrix}+C_2e^{-t}\begin{bmatrix}0\\1\end{bmatrix}$ ____________

(e) $C_1e^{t}\begin{bmatrix}2\cos t\\-\sin t\end{bmatrix}+C_2e^{t}\begin{bmatrix}2\sin t\\\cos t\end{bmatrix}$ ____________

4. (5 points) Consider the third order linear equation

$$y''' + 4y'' = 0.$$

What is its general solution?

(a) $y(t) = C_1 + C_2 \cos 2t + C_3 \sin 2t$

(b) $y(t) = C_1 + C_2 t + C_3 e^{-4t}$

(c) $y(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t}$

(d) $y(t) = C_1 + C_2 t + C_3 t^4$

5. (5 points) Evaluate the following definite integral

$$\int_0^\infty e^{-st} \delta(t - \frac{\pi}{2}) \cos t \, dt.$$

(Hint: The expression is equal to $\mathcal{L}\{\delta(t - \frac{\pi}{2}) \cos t\}$.)

(a) 0

(b) $e^{-\frac{\pi}{2}s} \frac{1}{s^2+1}$

(c) $e^{-\frac{\pi}{2}s} \frac{s}{s^2+1}$

(d) $e^{-\frac{\pi}{2}s} \frac{\pi}{2}$

6. (5 points) The inverse Laplace transform of $F(s) = \dfrac{2s-6}{s^2+4s+8}$ is

(a) $2e^{2t}\cos 2t - 4e^{2t}\sin 2t$,

(b) $2e^{2t}\cos 2t - 8e^{2t}\sin 2t$,

(c) $2e^{-2t}\cos 2t - 10e^{-2t}\sin 2t$,

(d) $2e^{-2t}\cos 2t - 5e^{-2t}\sin 2t$.

7. (5 points) Find the Laplace transform $\mathcal{L}\{u_1(t)(t^2-t)\}$.

(Recall: $u_c(t) = u(t-c)$.)

(a) $e^{-s}\left(\dfrac{2}{s^3} - \dfrac{1}{s^2}\right)$

(b) $e^{-s}\left(\dfrac{2}{s^3} + \dfrac{1}{s^2}\right)$

(c) $e^{-s}\left(\dfrac{2}{s^3} - \dfrac{1}{s^2} + \dfrac{1}{s}\right)$

(d) $e^{-s}\left(\dfrac{2}{s^3} - \dfrac{1}{s^2} - \dfrac{1}{s}\right)$

8. (5 points) Suppose $y(t)$ is the solution of the second order linear initial value problem

$$2y'' - y' + 2y = \delta(t-10), \qquad y(0) = 0, \quad y'(0) = 1.$$

What is $Y(s)$, the Laplace transform of $y(t)$?

(a) $Y(s) = \dfrac{e^{-10s} + 2}{2s^2 - s + 2}$

(b) $Y(s) = \dfrac{e^{-10s} + 1}{s^2 - s + 2}$

(c) $Y(s) = \dfrac{e^{-10s} - s}{s^2 - s + 2}$

(d) $Y(s) = \dfrac{e^{-10s} + 2s}{2s^2 - s + 2}$

9. (5 points) Suppose

$$f(t) = u_1(t) + 2u_{3/2}(t)\,t + 3u_3(t)\,t^2 + 4u_4(t)\,t^3.$$

What is $f(2)$?

(a) 3

(b) 5

(c) 17

(d) 49

10. (5 points) Suppose the linear system below has a (neutrally) stable center at $(0,0)$. What is/are the value(s) of γ?

$$\mathbf{x}' = \begin{bmatrix} -4 & 5 \\ -5 & \gamma \end{bmatrix} \mathbf{x}.$$

(a) 0

(b) 4

(c) $-2, 2$

(d) $-4, 4$

11. (5 points) Which system of first order linear equations below is equivalent to the second order linear equation

$$y'' - 7y' + 2y = 0?$$

(a) $\begin{cases} x_1' &= x_2 \\ x_2' &= 2x_1 + 7x_2 \end{cases}$

(b) $\begin{cases} x_1' &= x_2 \\ x_2' &= 2x_1 - 7x_2 \end{cases}$

(c) $\begin{cases} x_1' &= x_2 \\ x_2' &= -2x_1 + 7x_2 \end{cases}$

(d) $\begin{cases} x_1' &= x_2 \\ x_2' &= -2x_1 - 7x_2 \end{cases}$

12. (5 points) Which of the points below is **not** a critical point of the nonlinear system of equations

$$\begin{aligned} x' &= x^2 - y^2 \\ y' &= xy + x + y + 1 \end{aligned} \; ?$$

(a) $(1,1)$

(b) $(1,-1)$

(c) $(-1,1)$

(d) $(-1,-1)$

13. (5 points) Given that the point $(0,1)$ is a critical point of the nonlinear system of equations

$$\begin{aligned} x' &= x^2y - xy^2 \\ y' &= xy - x - 3y + 3 \end{aligned} \; .$$

This critical point $(0,1)$ is a(n)

(a) unstable saddle point.

(b) unstable spiral point.

(c) asymptotically stable node.

(d) asymptotically stable proper node (star point).

14. (15 points) True or false:

(a) (3 points) $\mathcal{L}\{f(t)-6g(t)\} = \mathcal{L}\{f(t)\} - 6\mathcal{L}\{g(t)\}$.

(b) (3 points) $\mathcal{L}\{4f(t)g(t)\} = 4\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.

(c) (3 points) $\mathcal{L}^{-1}\{e^{-2s}\dfrac{4}{(s+1)^5}\} = \dfrac{1}{6}u_2(t)t^4e^{-t}$.

(d) (3 points) Suppose $\mathbf{A}$ is a 2x2 matrix, and if the matrix-vector products $\mathbf{A}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, then the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has an asymptotically stable node at $(0,0)$.

(e) (3 points) The matrix $\begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$ is an example of a matrix that have two linearly independent eigenvectors.

15. (12 points)

(a) (9 points) Find the general solution of the homogeneous linear system

$$\mathbf{x}' = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix} \mathbf{x}.$$

(b) (3 points) Given that $\mathbf{x}(0) = \begin{bmatrix} \alpha \\ 5 \end{bmatrix}$, and $\lim_{t\to\infty} \mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, find the value of α.

16. (8 points) In the parts below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

   (a) (5 points) Suppose one of the eigenvalues of the coefficient matrix $\mathbf{A}$ is $r = 1 + 2i$, which has a corresponding eigenvector $\begin{bmatrix} 4 \\ 2 - 3i \end{bmatrix}$. Write down the system's <u>real-valued</u> general solution.

   (b) (3 points) Find the solution satisfying the initial condition $\mathbf{x}(0) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$.