

MATH 251
Midterm Exam II
April 1, 2004

Name: _____
Student Number: _____
Instructor: _____
Section: _____

This exam has 9 questions for a total of 100 points.

In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.

THE USE OF CALCULATORS IS NOT PERMITTED IN THIS EXAMINATION.

The last page of this examination contains a table of Laplace transforms for your use.

Do not write in this box.

1: _____
2: _____
3: _____
4: _____
5: _____
6: _____
7: _____
8: _____
9: _____
Total: _____

1. Circle the correct answer to each of the following statements:

- (a) (2 points) True or False: The Laplace transform of a piecewise continuous function $f(t)$ is defined by:

$$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st} f(t) dt$$

- (b) (2 points) True or False: A critically damped spring-mass system cannot oscillate, but the mass may cross its rest position as $t \rightarrow +\infty$.
- (c) (2 points) True or False: When choosing the form for a particular solution to a second order linear non-homogeneous equation, an extra multiplicative factor of t is always needed when the characteristic equation of the associated homogeneous problem has complex roots.
- (d) (2 points) True or False: The Laplace transform of a product of two functions is obtained by taking the product of the Laplace transforms of each function, i.e. $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$.

2. Consider the equation

$$2u'' + 32u = 9 \cos \omega t .$$

- (a) (2 points) Give a value of ω at which resonance will occur.
- (b) (2 points) If $\omega = 4.5$, what is the beat frequency (i.e., the frequency of observing successive maxima or successive minima in the solution $u(t)$)?

3. (12 points) Solve the initial value problem

$$y'' + 2y' - 3y = 3e^t; \quad y(0) = 0, \quad y'(0) = \frac{-5}{4}.$$

4. Consider a spring-mass system where the spring has a Hooke's constant of 2 N/m , and where the damping constant is $2 \text{ N}\cdot\text{s/m}$.
- (a) (3 points) Give a specific condition on a mass m , in kg, which guarantees that the system will be underdamped.
- (b) (5 points) For $m = 1 \text{ kg}$ and an initial displacement of 10 cm (i.e., $\frac{1}{10} \text{ m}$), set up and solve an initial value problem for the equation of motion if the mass is simply released at $t = 0$.
- (c) (2 points) What is the quasi-frequency of the motion you determined in (b)?
- (d) (2 points) If the same 1 kg mass were attached to a weaker spring with Hooke's constant $\frac{1}{3} \text{ N/m}$, would the system be overdamped, underdamped, or critically damped?

5. (15 points) Use the Laplace transform to solve the initial value problem

$$y'' - 2y' - 15y = e^{2t} + 2\delta(t - 1); \quad y(0) = 0, \quad y'(0) = 0.$$

6. Suppose an 8 lb bob causes a spring to stretch 6 in when hanging at rest. At $t = 0$, a force of 1 lb is applied in the downward direction for a duration of 1 s, and then removed. Assume there is no damping.
- (a) (5 points) Using step functions, derive an expression $f(t)$ for the external force described above as a function of time. You may use the downward direction as the positive coordinate direction.
- (b) (5 points) Using your result in (a) and the information provided in the problem, give an initial value problem modeling this system.
- (c) (5 points) Solve the initial value problem you found in (b).

7. (12 points) Find the Laplace transform of

$$f(t) = \begin{cases} 2t & 0 \leq t < 2; \\ t^2 & t \geq 2. \end{cases}$$

8. Find the inverse Laplace transform of each function:

(a) (8 points)

$$\frac{(s + 4)}{s^2 - 4s + 13}$$

(b) (8 points)

$$\frac{e^{-5s}}{s^2(s - 2)}$$

9. (6 points) Give an appropriate form for finding the particular solution to

$$y'' + 4y = 3te^t - 3\sin t + \cos 2t$$

as if you were going to use the method of undetermined coefficients. **Do not actually compute the coefficients.**