This exam has 11 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown.** For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

**PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.**

**YOU MAY NOT USE A CALCULATOR ON THIS EXAM.**
1. (5 points) Consider the function

\[
f(t) = \begin{cases} 
0, & t < 3 \\
t^2, & 3 \leq t < 5 \\
3 \cos(t), & t \geq 5
\end{cases}
\]

Which of the following expressions also describes \( f(t) \)?

(a) \( f(t) = (u_5(t) - u_3(t)) t^2 + u_5(t)3 \cos t \)

(b) \( f(t) = u_3(t)t^2 + u_5(t)3 \cos t \)

(c) \( f(t) = u_3(t)t^2 + u_5(t)(-t^2 + 3 \cos t) \)

(d) \( f(t) = (t^2 - 3 \cos t) u_3(t) + u_5(t)t^2 \)

2. (5 points) Find the Laplace transform \( \mathcal{L}\{u_1(t)e^{-3t}\sin(t - 1)\} \).

(a) \( F(s) = \frac{e^{3-s}}{(s-3)^2 + 1} \)

(b) \( F(s) = \frac{e^{-3-s}}{(s-3)^2 + 1} \)

(c) \( F(s) = \frac{e^{-3-s}}{(s+3)^2 + 1} \)

(d) \( F(s) = \frac{e^{3-s}}{(s+3)^2 + 1} \)
3. (5 points) Let $y(t)$ be the solution of the initial value problem

$$y'' - 2y' + 3y = u_2(t) - u_4(t), \quad y(0) = 0, \quad y'(0) = 8.$$ 

Find its Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$.

(a) $Y(s) = \frac{e^{-2s}}{s(s^2 - 2s + 3)} - \frac{e^{-4s}}{s^2 - 2s + 3} - \frac{8}{s^2 - 2s + 3}$

(b) $Y(s) = \frac{e^{-2s}}{s^2 - 2s + 3} - \frac{e^{-4s}}{s^2 - 2s + 3} - \frac{8s}{s^2 - 2s + 3}$

(c) $Y(s) = \frac{e^{-2s}}{s(s^2 - 2s + 3)} - \frac{e^{-4s}}{s^2 - 2s + 3} + \frac{8}{s^2 - 2s + 3}$

(d) $Y(s) = \frac{e^{-2s}}{s(s^2 - 2s + 3)} - \frac{e^{-4s}}{s^2 - 2s + 3} + \frac{8s}{s^2 - 2s + 3}$

4. (5 points) Consider the system of linear equations

$$\begin{align*}
x' &= y \\
y' &= 5x - 2y + 3t^2
\end{align*}$$

Which of the following equations is equivalent to this system?

(a) $u'' + 2u' - 5u = 3t^2$

(b) $u'' - 2u' + 5u = 3t^2$

(c) $u'' + 5u' - 2u = 3t^2$

(d) $u'' - 5u' + 2u = 3t^2$
5. (5 points) Consider a certain $2 \times 2$ linear system $\mathbf{x}' = A\mathbf{x}$, where $A$ is a matrix of real numbers. Suppose all of its nonzero solutions are bounded, but they do not reach a limit as $t \to +\infty$. Which of the following is a possible pair of eigenvalues of $A$?

(a) $-1, -4$
(b) $-1 \pm 4i$
(c) $\pm i$
(d) $-1, 4$

6. (5 points) Consider the system of linear equations

$$\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}.$$ 

Which of the following statements is TRUE?

(a) The critical point at $(0, 0)$ is a node.
(b) The critical point at $(0, 0)$ is a saddle point.
(c) Its general solution is $\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
(d) Its general solution is $\mathbf{x}(t) = C_1 \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$. 
7. (18 points) For each part below, determine whether the statement is true or false. To receive credit you must (briefly) justify each answer.

(a) Suppose \( L\{f(t)\} = \frac{s^2}{s^3 + 6} \), then \( L\{e^{2t}f(t)\} = \frac{(s + 2)^2}{(s + 2)^3 + 6} \).

(b) Suppose \( f(t) = tu_2(t) - u_4(t)(t + 1) \), then \( f(3) = 3 \).

(c) Suppose \( f \) is a continuous function such that \( f(4) = 0 \), then \( L\{\delta(t - 4)e^{f(t)}\} = e^{-4s} \).

(d) Some, but not all, nonzero solutions of \( x' = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} x \) diverge to infinity as \( t \to +\infty \).

(e) The system \( x' = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \) has a critical point at \((-1, 2)\) which is a saddle point.

(f) The nonlinear system \( \begin{cases} x' = y \cdot F(x, y) \\ y' = x \cdot G(x, y) \end{cases} \), where \( F \) and \( G \) have continuous partial derivatives everywhere, always has a critical point at \((0, 0)\).
8. (15 points)

(a) (9 points) Find the inverse Laplace transform of:

\[ F(s) = e^{-6s} \frac{5s + 20}{s^3 + 2s^2 + 10s}. \]

In parts (b) and (c), suppose that \( \mathcal{L}\{f(t)\} = \frac{s^2}{s^3 + 6} \), and that \( f(0) = -1, f'(0) = 2 \).

(b) (3 points) Determine \( \mathcal{L}\{tf(t)\} \).

(c) (3 points) Determine \( \mathcal{L}\{f''(t)\} \).
9. (14 points) Use the Laplace transform to solve the following initial value problem.

\[ y'' - 2y' - 3y = 12u_6(t) + 4\delta(t - 2), \quad y(0) = 0, \quad y'(0) = 10. \]
10. (12 points) Consider the system of linear equations
\[ \mathbf{x}' = \begin{bmatrix} 1 & -4 \\ -1 & 1 \end{bmatrix} \mathbf{x}. \]

(a) (6 points) Find the general solution of this system.

(b) (2 points) Classify the type and stability of the critical point at \((0, 0)\).

(c) (4 points) Given the initial condition \(\mathbf{x}(0) = \begin{bmatrix} 4 \\ \beta \end{bmatrix}\), and suppose \(\lim_{t \to \infty} \mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\), determine the value of \(\beta\).
11. (11 points) Consider the autonomous nonlinear system:

\[
\begin{align*}
    x' &= x(y - x + 1) \\
    y' &= y(x - 2y + 1)
\end{align*}
\]

(a) (4 points) The system has 4 critical points. One of the critical points is \((0, \frac{1}{2})\). Verify that \((0, \frac{1}{2})\) is indeed a critical point. Then find the other 3 critical points.

(b) (7 points) Linearize the system about the point \((0, \frac{1}{2})\). Classify the type and stability of this critical point by examining the linearized system. Be sure to clearly state the linearized system’s coefficient matrix and its eigenvalues.
<table>
<thead>
<tr>
<th>$f(t) = \mathcal{L}^{-1}{F(s)}$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
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<tbody>
<tr>
<td>1. $1$</td>
<td>$\frac{1}{s}$</td>
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<tr>
<td>2. $e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
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<tr>
<td>3. $t^n, \ n = \text{positive integer}$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
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<td>4. $t^p, \ p &gt; -1$</td>
<td>$\frac{\Gamma(p + 1)}{s^{p+1}}$</td>
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<td>5. $\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
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<td>6. $\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
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<td>7. $\sinh at$</td>
<td>$\frac{a}{s^2 - a^2}$</td>
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<td>8. $\cosh at$</td>
<td>$\frac{s}{s^2 - a^2}$</td>
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<td>9. $e^{at} \sin bt$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$</td>
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<td>10. $e^{at} \cos bt$</td>
<td>$\frac{s-a}{(s-a)^2 + b^2}$</td>
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<tr>
<td>11. $t^n e^{at}, \ n = \text{positive integer}$</td>
<td>$\frac{n!}{(s-a)^{n+1}}$</td>
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<td>12. $u_c(t)$</td>
<td>$e^{-cs} \frac{e^{-cs}}{s}$</td>
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<tr>
<td>13. $u_c(t) f(t - c)$</td>
<td>$e^{-cs} F(s)$</td>
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<td>14. $e^{ct} f(t)$</td>
<td>$F(s - c)$</td>
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<td>15. $f(ct)$</td>
<td>$\frac{1}{c} F\left(\frac{s}{c}\right)$</td>
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<tr>
<td>16. $(f * g)(t) = \int_0^t f(t - \tau)g(\tau) , d\tau$</td>
<td>$F(s)G(s)$</td>
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<tr>
<td>17. $\delta(t - c)$</td>
<td>$e^{-cs}$</td>
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<td>18. $f^{(n)}(t)$</td>
<td>$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$</td>
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<tr>
<td>19. $(-t)^n f(t)$</td>
<td>$F^{(n)}(s)$</td>
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