MATH 251

Examination II

November 9, 2015

FORM A

Name:________________

Student Number:________________

Section:________________

This exam has 12 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE AND ALL OTHER MOBILE DEVICES.

Do not write in this box.

1 through 7:		(35)
8:		(14)
9:		(15)
10:		(14)
11:		(12)
12:		(10)
Total:		

1. (5 points) Evaluate the following definite integral

$$\int_0^\infty e^{-(s+4)t} \cos(2t)\, dt.$$

(a) $\dfrac{s+4}{(s+4)^2+4}$

(b) $\dfrac{s}{(s+4)(s^2+4)}$

(c) $\dfrac{s-4}{(s-4)^2+4}$

(d) $e^{-4s}\dfrac{s}{s^2+4}$

2. (5 points) Find the Laplace transform $\mathcal{L}\{u_4(t)(16-t^2)\}$.

(a) $F(s) = e^{-4s}\left(\dfrac{16}{s} - \dfrac{2}{s^3}\right)$

(b) $F(s) = e^{-4s}\left(-\dfrac{8}{s^2} - \dfrac{2}{s^3}\right)$

(c) $F(s) = e^{-4s}\left(\dfrac{12}{s} - \dfrac{2}{s^3}\right)$

(d) $F(s) = e^{-4s}\left(\dfrac{16}{s^2} - \dfrac{2}{s^4}\right)$

3. (5 points) Find the inverse Laplace transform of $F(s) = \dfrac{3s-3}{s^2+2s+10}$.

(a) $f(t) = 3e^{-t}\cos(3t)$

(b) $f(t) = 3e^{-t}\cos(3t) - e^{-t}\sin(3t)$

(c) $f(t) = 3e^{-t}\cos(3t) - 2e^{-t}\sin(3t)$

(d) $f(t) = 3e^{-t}\cos(3t) - 3e^{-t}\sin(3t)$

4. (5 points) Let $y(t)$ be the solution of the initial value problem

$$y'' - 6y' = 2 + u_\pi(t), \qquad y(0) = 1, \qquad y'(0) = -1.$$

Find its Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$.

(a) $Y(s) = \dfrac{e^{-\pi s} - s^2 + 7s + 2}{s^3 - 6s^2}$

(b) $Y(s) = \dfrac{e^{-\pi s} + s^2 + 5s + 2}{s^3 - 6s^2}$

(c) $Y(s) = \dfrac{e^{-\pi s} + s^2 + 2}{s^3 - 6s^2}$

(d) $Y(s) = \dfrac{e^{-\pi s} + s^2 - 7s + 2}{s^3 - 6s^2}$

5. (5 points) Which second order differential equation below is equivalent to the following system?

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$$

(a) $y'' - 3y' + y = e^{2t}$

(b) $y'' - 3y' - y = -e^{2t}$

(c) $y'' + 3y' - y = e^{2t}$

(d) $y'' + 3y' + y = -e^{2t}$

6. (5 points) Suppose a certain linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 matrix of real numbers with nonzero eigenvalues, has at least one nonzero solution that reaches a finite limit as $t \to \infty$ **or** $t \to -\infty$. Which statement(s) below regarding the critical point $(0, 0)$ could possibly be true?

I It is asymptotically stable.

II It is (neutrally) stable.

III It is a saddle point.

(a) I and II

(b) I and III

(c) II and III

(d) I, II, and III

7. (5 points) Consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of real numbers. Suppose one of the eigenvalues of the coefficient matrix $\mathbf{A}$ is $r = 1 - i$, which has a corresponding eigenvector $\begin{bmatrix} 2+3i \\ 1 \end{bmatrix}$. What is the system's real-valued general solution?

(a) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} 2\cos t - 3\sin t \\ \cos t \end{bmatrix} + C_2 e^t \begin{bmatrix} 2\sin t - 3\cos t \\ -\sin t \end{bmatrix}$

(b) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} 2\cos t + 3\sin t \\ \cos t \end{bmatrix} + C_2 e^t \begin{bmatrix} 2\sin t - 3\cos t \\ \sin t \end{bmatrix}$

(c) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} 2\cos t + 3\sin t \\ \sin t \end{bmatrix} + C_2 e^t \begin{bmatrix} 2\cos t - 3\sin t \\ \cos t \end{bmatrix}$

(d) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} 2\cos t - 3\sin t \\ \cos t \end{bmatrix} + C_2 e^t \begin{bmatrix} 2\sin t + 3\cos t \\ \sin t \end{bmatrix}$

8. (14 points) Determine whether each statement below is true or false. You must justify your answers.

(a) (2 points) $\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+1)}\} = t\,\sin(t)$

(b) (2 points) $\mathcal{L}\{t(t-1)e^{2t-3}\} = e^{-3}\mathcal{L}\{t^2 e^{2t}\} - e^{-3}\mathcal{L}\{t\,e^{2t}\}$.

(c) (2 points) The linear system $\mathbf{x}' = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} \mathbf{x}$ has a critical point at $(0, 0)$ that is an asymptotically stable proper node.

(d) (2 points) Depending on the initial condition, the origin $(0, 0)$ might not be a critical point of a homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(e) (2 points) The origin $(0, 0)$ is always a critical point of the nonlinear system $\begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases}$.

(f) (2 points) Suppose a 2×2 matrix $\mathbf{A}$ satisfies the relation $\mathbf{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$, then $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-3t}$ is a solution to the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(g) (2 points) If a 2×2 matrix $\mathbf{A}$ has two complex eigenvalues $-1 \pm \beta i$, where $\beta > 0$, then the origin as the equilibrium solution of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is asymptotically stable regardless of what the value β is.

9. (15 points) Suppose $\mathcal{L}\{f(t)\} = \dfrac{s}{s^4+10}$, and that $f(0) = 1$, $f(1) = 6$, $f(2) = -1$, $f(3) = -4$. Answer each question below.

(a) Determine $\mathcal{L}\{t\, f(t)\}$.

(b) Determine $\mathcal{L}\{f'(t)\}$.

(c) Determine $\mathcal{L}\{e^{-3t}\, f(t)\}$.

(d) Determine $\mathcal{L}\{\delta(t-2) f(t)\}$.

(e) Let $y(t) = \mathcal{L}^{-1}\{e^{-2s}\dfrac{s}{s^4+10}\}$. Evaluate $y(1)$.

10. (14 points) Use the Laplace transform to solve the following initial value problem.

$$y'' + 4y = t + 3\delta(t-2), \qquad y(0) = 1, \qquad y'(0) = 0.$$

11. (12 points)

(a) (8 points) Find the general solution of the system of linear equations

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix} \mathbf{x}.$$

(b) (2 points) Find the solution satisfying $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) (2 points) Classify the type and stability of the critical point at $(0, 0)$.

12. (10 points) Consider the autonomous nonlinear system:

$$\begin{aligned} x' &= 2x - x^2 - xy \\ y' &= (1-y)(2+x) \end{aligned}$$

(a) (3 points) The system has 3 critical points. One of the critical points is $(0, 1)$. Verify that $(0, 1)$ is indeed a critical point. Then find the other 2 critical points.

(b) (7 points) Linearize the system about the point $(0, 1)$. Classify the type and stability of this critical point by examining the linearized system. Be sure to clearly state the linearized system's coefficient matrix and its eigenvalues.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\frac{a}{s^2+a^2}$
6.	$\cos at$	$\frac{s}{s^2+a^2}$
7.	$\sinh at$	$\frac{a}{s^2-a^2}$
8.	$\cosh at$	$\frac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
16.	$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^nf(t)$	$F^{(n)}(s)$