

MATH 251

Examination II

November 8, 2011

FORM A

Name: _____

Student Number: _____

Section: _____

This exam has 12 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE AND ALL OTHER MOBILE DEVICES.

Do not write in this box.

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6: _____
7: _____
8: _____
9: _____
10: _____
11: _____
12: _____
Total: _____

1. (5 points) Consider the fourth order linear equation

$$y^{(4)} - 8y'' + 16y = 0.$$

What is its general solution?

- (a) $y(t) = C_1 \cos 2t + C_2 \sin 2t + C_3 t \cos 2t + C_4 t \sin 2t$
- (b) $y(t) = C_1 + C_2 t + C_3 \cos 4t + C_4 \sin 4t$
- (c) $y(t) = C_1 e^{2t} + C_2 e^{-2t} + C_3 t e^{2t} + C_4 t e^{-2t}$
- (d) $y(t) = C_1 + C_2 t + C_3 e^{4t} + C_4 t e^{4t}$

2. (5 points) Suppose

$$f(t) = 1 + u_1(t) t - 2u_2(t) t^2 + 5u_5(t) t^3.$$

Find $f(2)$.

- (a) 3
- (b) -5
- (c) 0
- (d) 35

3. (5 points) Find the Laplace transform $\mathcal{L}\{u_3(t)(t^2 - 3)\}$.

(a) $F(s) = e^{-3s} \frac{6s^2 + 6s + 2}{s^3}$

(b) $F(s) = e^{-3s} \frac{2}{s^4}$

(c) $F(s) = e^{-3s} \frac{12s^2 - 6s + 2}{s^4}$

(d) $F(s) = e^{-3s} \frac{2 - 6s^2}{s^3}$

4. (5 points) Suppose

$$\mathcal{L}\{f(t)\} = \frac{s^2}{s^3 + 27}.$$

What is $\mathcal{L}\{7e^{5t}f(t)\}$?

(a) $\frac{7(s-5)^2}{(s-5)^3 + 27}$

(b) $\frac{7s^2}{(s-5)(s^3 + 27)}$

(c) $\frac{7s^2}{s(s-5)(s^3 + 27)}$

(d) $\frac{7(s+5)^2}{(s+5)^3 + 27}$

5. (5 points) Which system of first order linear equations below is equivalent to the third order linear equation

$$y''' + y'' - 3y' - y = 0?$$

(a)
$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_1 - 3x_2 - x_3 \end{cases}$$

(b)
$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -x_1 + 3x_2 + x_3 \end{cases}$$

(c)
$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -x_1 - 3x_2 + x_3 \end{cases}$$

(d)
$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_1 + 3x_2 - x_3 \end{cases}$$

6. (5 points) Consider a certain linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where \mathbf{A} is a matrix of real numbers with complex conjugate eigenvalues. Suppose some of its solutions do not reach a limit either as $t \rightarrow +\infty$, or as $t \rightarrow -\infty$. Then the critical point $(0, 0)$ must be a(n)

- (a) unstable spiral point.
- (b) unstable saddle point.
- (c) asymptotically stable improper node.
- (d) (neutrally) stable center.

7. (12 points) Consider various mass-spring systems and the differential equations that describe their displacement. A list of equations is given below. Each equation may or may not describe the displacement of any mass-spring system.

A. $y'' + 7y' + 6y = 0$

B. $y'' + 9y = 3 \sin 9t$

C. $y'' + y = 0$

D. $y'' - 4y' - 8y = 0$

E. $y'' + 6y' + 9y = 0$

F. $y'' + 4y = -4 \cos 2t$

G. $y'' + 2y' + 5y = 0$

For each of parts (a) through (d) below, write down the letter corresponding to the equation on the list above describing the correct mass-spring system with the specified behavior. There is only one correct equation to each part. However, an equation may be re-used for more than one part.

(a) (3 points) This system is critically damped.

(b) (3 points) This system is undergoing resonance.

(c) (3 points) This system has a natural period of π seconds.

(d) (3 points) This system has solutions that oscillate at a quasi-frequency of 2 radians per second.

8. (10 points) For each part below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where \mathbf{A} is a 2×2 matrix of real numbers. Based solely on the information given in each part, determine the type and stability of the system's critical point at $(0, 0)$.

(a) The coefficient matrix is $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

(b) The coefficient matrix is $\mathbf{A} = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{7} \end{bmatrix}$.

(c) Eigenvalues of \mathbf{A} are $-5 + 2i$ and $-5 - 2i$.

(d) One of the eigenvalues of \mathbf{A} is $-4i$.

(e) The general solution is $\mathbf{x}(t) = C_1 e^{-t/19} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-t/19} \begin{bmatrix} 2t \\ 5 + t \end{bmatrix}$.

9. (14 points) Find the inverse Laplace transform of each function given below.

(a) (7 points)
$$F(s) = \frac{6}{s^3 + 9s}$$

(b) (7 points)
$$F(s) = e^{-s} \frac{2s - 4}{s^2 + 2s + 10}$$

10. (14 points) Use the Laplace transform to solve the following initial value problem.

$$y'' + 2y' + y = 2u_5(t) - \delta(t - 10), \quad y(0) = 0, \quad y'(0) = -4.$$

11. (10 points) Consider the initial value problem.

$$\mathbf{x}' = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ -10 \end{bmatrix}.$$

(a) (8 points) Solve the initial value problem.

(b) (2 points) Classify the type and stability of the critical point at $(0, 0)$.

12. (10 points) Consider the nonlinear system:

$$\begin{aligned}x' &= x^2 - xy \\y' &= xy - 3x + 2\end{aligned}$$

(a) (2 points) One of the critical points of the system is $(1, 1)$. There is another critical point. Find it.

(b) (8 points) Linearize the system about the point $(1, 1)$. Classify the type and stability of the critical point at $(1, 1)$ by examining the linearized system. Be sure to clearly state the linearized system's matrix and its eigenvalues.