MATH 251

Examination II

November 8, 2010

FORM A

Name: ____________________

Student Number: ____________________

Section: ____________________

This exam has 12 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM.

Do not write in this box.

1 thru 6: ____________

7: ____________

8: ____________

9: ____________

10: ____________

11: ____________

12: ____________

Total: ____________

1. (5 points) Consider the nonhomogeneous second order linear equation

$$y'' + 25y = 3e^{5t} - 2t\sin 5t.$$

Which function below is the most suitable choice of the **form** of particular solution Y that you would use to solve the given equation using the Method of Undetermined Coefficients?

(a) $Y = Ae^{5t} + (Bt + C)\cos 5t + (Dt + E)\sin 5t$

(b) $Y = Ate^{5t} + (Bt + C)\cos 5t + (Dt + E)\sin 5t$

(c) $Y = Ae^{5t} + (Bt^2 + Ct)\cos 5t + (Dt^2 + Et)\sin 5t$

(d) $Y = Ate^{5t} + (Bt^2 + Ct)\cos 5t + (Dt^2 + Et)\sin 5t$

2. (5 points) Consider the fourth order linear equation

$$y^{(4)} + 9y'' = 0.$$

What is its general solution?

(a) $y(t) = C_1 e^t + C_2 e^{-t} + C_3\cos 3t + C_4\sin 3t$

(b) $y(t) = C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t + C_3\, t\cos\sqrt{3}t + C_4\, t\sin\sqrt{3}t$

(c) $y(t) = C_1\cos t + C_2\sin t + C_3\cos 3t + C_4\sin 3t$

(d) $y(t) = C_1 + C_2\, t + C_3\cos 3t + C_4\sin 3t$

3. (5 points) Find the Laplace transform $\mathcal{L}\{u_4(t)\,(t-2)^2\}$.

(a) $F(s) = e^{-4s}\,\dfrac{4s^2+4s+2}{s^3}$

(b) $F(s) = e^{-4s}\,\dfrac{4s^2-4s+2}{s^4}$

(c) $F(s) = e^{-4s}\,\dfrac{4s^2-4s+2}{s^3}$

(d) $F(s) = e^{-4s}\,\dfrac{36s^2-12s+2}{s^4}$

4. (5 points) Evaluate the following definite integral

$$\int_0^\infty e^{-(s+1)t}\sin(2t)\,dt.$$

(Hint: This integral represents the Laplace transform of a certain function. It is absolutely not necessary to integrate in order to find the answer.)

(a) $\dfrac{2}{s^2+2s+5}$

(b) $e^{-s}\,\dfrac{2e^{-1}}{s^2+4}$

(c) $e^{-s}\,\dfrac{s}{s^2+4}$

(d) $\dfrac{1}{s^2-2s+5}$

5. (5 points) Which system of first order linear equations below is equivalent to the second order linear equation

$$y'' - 5y' + 6y = t^2 - t?$$

(a) $\begin{cases} x_1' &= x_2 \\ x_2' &= -5x_1 + 6x_2 + t^2 - t \end{cases}$

(b) $\begin{cases} x_1' &= x_2 \\ x_2' &= 5x_1 - 6x_2 - t^2 + t \end{cases}$

(c) $\begin{cases} x_1' &= x_2 \\ x_2' &= -6x_1 + 5x_2 + t^2 - t \end{cases}$

(d) $\begin{cases} x_1' &= x_2 \\ x_2' &= 6x_1 - 5x_2 - t^2 + t \end{cases}$

6. (5 points) Consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix of real numbers. Suppose one of the eigenvalues of the coefficient matrix $\mathbf{A}$ is $r = 1 + 4i$, which has a corresponding eigenvector $\begin{bmatrix} -3 - 7i \\ 2 \end{bmatrix}$. What is the system's real-valued general solution?

(a) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} -3\cos 4t - 7\sin 4t \\ 2\cos 4t \end{bmatrix} + C_2 e^t \begin{bmatrix} -7\cos 4t + 3\sin 4t \\ 2\sin 4t \end{bmatrix}$

(b) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} -3\cos 4t + 7\sin 4t \\ 2\cos 4t \end{bmatrix} + C_2 e^t \begin{bmatrix} -7\cos 4t - 3\sin 4t \\ 2\sin 4t \end{bmatrix}$

(c) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} -3\cos 4t + 7\sin 4t \\ -2\sin 4t \end{bmatrix} + C_2 e^t \begin{bmatrix} 7\cos 4t - 3\sin 4t \\ 2\cos 4t \end{bmatrix}$

(d) $\mathbf{x}(t) = C_1 e^t \begin{bmatrix} -3\cos 4t - 7\sin 4t \\ 2\sin 4t \end{bmatrix} + C_2 e^t \begin{bmatrix} 7\cos 4t + 3\sin 4t \\ 2\cos 4t \end{bmatrix}$

7. (8 points) Consider various mass-spring systems and the differential equations that describe their displacement. A list of equations is given below. Each equation may or may not describe the displacement of any mass-spring system.

A. $y'' + 4y' + 5y = 0$

B. $y'' + 4y' + 3y = 0$

C. $y'' - y = \cos t$

D. $y'' + 9y = 0$

E. $y'' - y' - 2y = 0$

F. $y'' + 2y' + y = 0$

G. $y'' + 16y = 2\sin 4t$

For each of parts (a) through (d) below, write down the letter corresponding to the equation on the list above describing the correct mass-spring system with the specified behavior. There is only one correct equation to each part. However, an equation may be re-used for more than one part.

(a) (2 points) This system is critically damped.

(b) (2 points) This system is undergoing resonance.

(c) (2 points) This system has solutions that oscillate freely with a constant amplitude.

(d) (2 points) This system has solutions that oscillate freely with an exponentially decreasing amplitude.

8. (10 points) Determine the type and stability of the critical point at $(0,0)$ for each of the 2x2 linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whose general solutions are given below. For the type, give the actual name. For the stability, use the letter **A** if the point is asymptotically stable, **U** if it is unstable, **S** if it is (neutrally) stable.

	Type	**Stability**
(a) $C_1e^{2t}\begin{bmatrix}\cos t\\ 2\sin t\end{bmatrix}+C_2e^{2t}\begin{bmatrix}2\sin t\\ -\cos t\end{bmatrix}$	__________	__________
(b) $C_1e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}+C_2e^{-t}\begin{bmatrix}-1\\ 1\end{bmatrix}$	__________	__________
(c) $C_1e^{\sqrt{5}t}\begin{bmatrix}1\\ 1\end{bmatrix}+C_2e^{\sqrt{7}t}\begin{bmatrix}-1\\ 1\end{bmatrix}$	__________	__________
(d) $C_1\begin{bmatrix}2\cos t\\ \sin t\end{bmatrix}+C_2\begin{bmatrix}\sin t\\ -2\cos t\end{bmatrix}$	__________	__________
(e) $C_1e^{\sqrt{2}t}\begin{bmatrix}1\\ 1\end{bmatrix}+C_2e^{\sqrt{2}t}\begin{bmatrix}t+1\\ t-1\end{bmatrix}$	__________	__________

9. (14 points) Find the inverse Laplace transform of each function given below.

(a) (7 points) $$F(s) = \frac{s^2+1}{s(s+1)^2}$$

(b) (7 points) $$F(s) = e^{-9s}\frac{-s+3}{s^2+6s+25}$$

10. (16 points) Use the Laplace transform to solve the following initial value problem.

$$y'' + 4y' + 3y = \delta(t) - u_6(t), \qquad y(0) = 2, \qquad y'(0) = 0.$$

No credit will be given if the Laplace transform is not used to solve this problem.

11. (12 points) Consider the system of linear equations

$$\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 7 & -5 \end{bmatrix} \mathbf{x}.$$

(a) (8 points) Find the general solution of this system.

(b) (2 points) Classify the type and stability of the critical point at $(0, 0)$.

(c) (2 points) Given that $\mathbf{x}(0) = \begin{bmatrix} -4 \\ \beta \end{bmatrix}$, and $\lim_{t\to\infty} \mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, find the value of β.

12. (10 points) Consider the nonlinear system:

$$\begin{aligned} x' &= x - y \\ y' &= 4y - x^2 y \end{aligned}$$

(a) (2 points) One of the critical points of the system is $(2,2)$. Verify that $(2,2)$ is indeed a critical point. That is, show that $(2,2)$ satisfies the condition(s) of being a critical point.

(b) (2 points) Besides $(2,2)$, there are 2 other critical points. Find those other 2 critical points of the system.

(c) (6 points) Linearize the system about the point $(2,2)$. Classify the type and stability of the critical point at $(2,2)$ by examining the linearized system. Be sure to clearly state the linearized system's matrix and its eigenvalues.