MATH 251

Examination II

November 12, 2007

Name: __________

Student Number: __________

Section: __________

This exam has 11 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number.

YOU MAY NOT USE A CALCULATOR ON THIS EXAM. PLEASE TURN OFF AND PUT AWAY YOUR CELL PHONE.

Do not write in this box.

1: __________
2: __________
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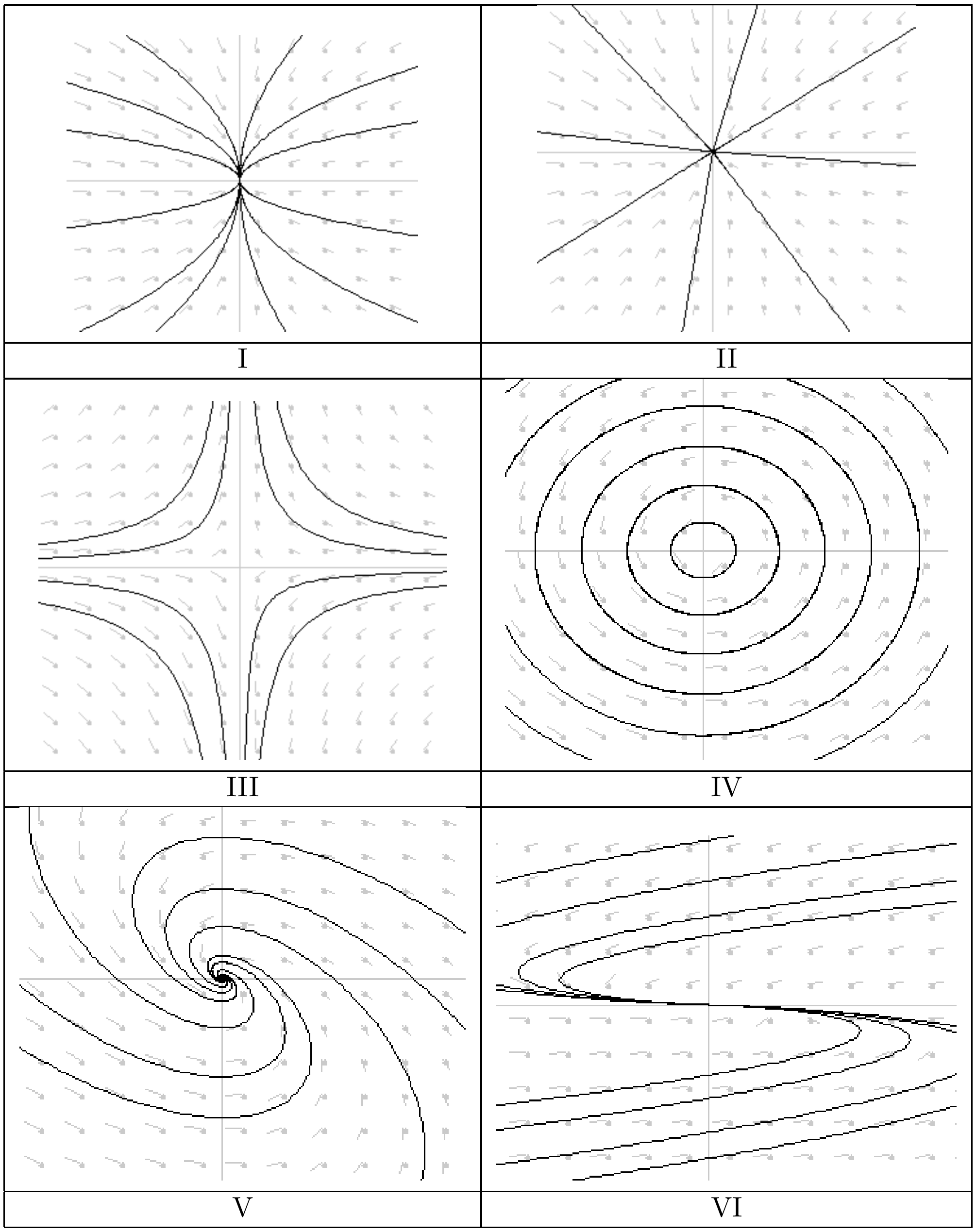
11: __________
Total: __________

1. (6 points) Match the sketches of phase portraits for 2x2 homogeneous linear systems with constant coefficients $\mathbf{x}' = A\mathbf{x}$ with the names of their critical points at the origin.



(a) saddle ______
(b) node ______
(c) proper node ______
(d) center ______
(e) spiral ______
(f) improper node ______

2. (5 points) When an object with mass 5 kg is attached to a spring, the object stretches the spring by 2 m. A damper with damping coefficient of 4 N/m is attached to the system. Assume there is no external force acting on the system and that acceleration due to gravity is 10 m/s^2. If the object is released 1 m above its equilibrium position and is given an initial downward velocity of 3 m/s, which initial value problem describes the displacement of the mass from its equilibrium position? Take the downward direction to be positive for all displacements and forces.

   (a) $5y'' + 4y' + 25y = 0, \quad y(0) = 1, \quad y'(0) = -3$
   (b) $5y'' + 4y' + 25y = 0, \quad y(0) = -1, \quad y'(0) = 3$
   (c) $5y'' + 4y' + 10y = 0, \quad y(0) = 1, \quad y'(0) = -3$
   (d) $5y'' + 4y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 3$

3. (5 points) Which of the following functions has $\int_0^\infty e^{-st}t^2\,dt$ as its Laplace transform?

   (a) t^2
   (b) $e^{-t}t^2$
   (c) t
   (d) $\delta(t-2)t^2$

4. (5 points) If $f(t) = 1 - u(t-1) + 4tu(t-2) - 10(t^3 - t)u(t-5)$, where $u(t-c) = u_c(t)$ is the unit step function, what is $f(3)$?

   (a) 1
   (b) 0
   (c) 12
   (d) -212

5. (5 points) Which of the following 2nd order equations is equivalent to the given linear system?

$$x_1' = x_2$$
$$x_2' = 3x_1 - 2x_2$$

(a) $y'' - 3y' + 2y = 0$
(b) $y'' + 2y' - 3y = 0$
(c) $y'' - 3y' + y = 0$
(d) $y'' + 5y' + y = 0$

6. A spring-mass system is modeled by the initial value problem

$$2y'' + \gamma y' + 8y = F(t), \quad \gamma \geq 0, \quad y(0) = 3, \quad y'(0) = -4.$$

(a) (6 points) If $\gamma = 0$ and $F(t) = 0$, what is the amplitude of displacement?

(b) (3 points) If $\gamma = 0$ and $F(t) = 3\cos(\omega t)$, for which value(s) of ω will the system undergo resonance?

(c) (4 points) If $F(t) = 0$, for which value(s) of γ will the system ***not*** oscillate?

7. (a) (6 points) Compute the following Laplace transform:

$$\mathcal{L}\{tu(t-3)\}$$

(b) (7 points) Compute the following inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$$

8. (14 points) Use Laplace transforms to solve the following initial value problem:

$$y' + 2y = e^{t-3}u(t-3) + \delta(t-3), \quad y(0) = 3.$$

Recall that $u(t-c) = u_c(t)$ is the unit step function.

9. Consider the 2x2 linear homogeneous system with constant coefficients $\mathbf{x}' = A\mathbf{x}$.

(a) (4 points) If the eigenvalues of A are $\pm i$, classify the type and stability of the critical point$(0,0)$.

(b) (5 points) If in addition an eigenvector corresponding to the eigenvalue i is $\begin{pmatrix} 2 \\ i \end{pmatrix}$, then write down the ***real-valued*** general solution $\mathbf{x}(t)$.

10. Consider the system

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix} \mathbf{x}.$$

(a) (10 points) Find the general solution $\mathbf{x}(t)$.

(b) (3 points) If $\mathbf{x}(0) = \begin{pmatrix} 6 \\ \beta \end{pmatrix}$ and $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$, what is β?

11. (a) (4 points) Find the critical points of the following nonlinear system:

$$x' = y(1 - x^2)$$
$$y' = x + y$$

(b) (8 points) Linearize the system

$$x' = 1 - y$$
$$y' = x^2 - y^2$$

around its critical point (1,1) and classify its type and stability.