MATH 251

Examination II

July 30, 2007

Name:________________

Student Number:________________

Section:________________

This exam has 9 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

Please turn off and put away your cell phone.

You may not use a calculator on this exam.

Do not write in this box.

1:	
2:	
3:	
4:	
5:	
6:	
7:	
8:	
9:	
Total:	

1. (14 points) Consider a mass-spring system described by the equation

$$3u'' + 12u' + ku = 0, \qquad k > 0.$$

Answer the following questions. Be sure to justify your answer. Full credit will not be given without supporting work.

(a) For what value(s) of k would the system be *overdamped*?

(b) When $k = 12$, determine whether the system is over-, under-, or critically damped.

(c) The system would oscillate when: $k = 9$ or $k = 15$.
(Circle the correct value and justify your answer.)

(d) Find the *quasi-period* of the system whose k-value you found in part (c).

(e) True or False: When $k = 3$ the mass will *never* cross the system's equilibrium position *more than once.*

2. (5 points) A mass of 2 kg stretches a spring 0.4 m. The system has no damping. At $t = 0$, the mass is pulled down 1 m from its equilibrium position and set in motion with an initial downward velocity of 4 m/s. You may use $g = 10\ m/s^2$ as the gravitational constant.

   (a) Find the Hooke's constant, k, of the spring.

   (b) Set up, **but do not solve**, an initial value problem to find the system's diplacement function $u(t)$.

3. (5 points) Given that $\mathcal{L}\{f(t)\} = F(s)$, what is $\mathcal{L}\{e^{5t} t f(t)\}$?

   (a) $\dfrac{1}{s^2(s-5)} F(s)$

   (b) $\dfrac{-1}{s-5} F'(s)$

   (c) $\dfrac{1}{s^2} F(s+5)$

   (d) $-F'(s-5)$

4. (14 points) Find the inverse Laplace transforms of

(a) $\dfrac{s^2 - 2s - 1}{s(s+1)^2}$

(b) $e^{-4s}\dfrac{2s - 8}{s^2 + 2s + 26}$

5. (14 points) Rewrite the following piecewise continuous function $f(t)$ in terms of the unit-step functions. Then find its Laplace transform $\mathcal{L}\{f(t)\}$.

$$f(t) = \begin{cases} e^{2t}, & 0 \le t < 4 \\ 8t - t^2, & 4 \le t \end{cases}$$

6. (14 points) Use the Laplace transform to solve the initial value problem

$$y'' + 5y' - 6y = \delta(t-3), \qquad y(0) = 0, \qquad y'(0) = 2.$$

No credit will be given if the Laplace transform is not used to solve this problem.

7. (8 points) Consider the initial value problem

$$y''' - 2y'' + y' + 4y = e^{-2t} + \delta(t - \pi), \qquad y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0.$$

Find $\mathcal{L}\{y(t)\} = Y(s)$, the Laplace transform of its solution. You do not need to simplify your answer. **Do <u>not</u> solve for its inverse transform**, $y(t)$!

8. (a) (14 points) Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 2 & 6 \\ 4 & 4 \end{pmatrix} \mathbf{x}.$$

(b) Given the initial condition $\mathbf{x}(0) = \begin{pmatrix} 6 \\ \beta \end{pmatrix}$, and suppose that $\lim_{t\to\infty} \mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find the value(s) of β.

9. (12 points) In each part below, consider a certain system of two first order linear differential equations in two unknowns, $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) Suppose that the system's general solution is

$$\mathbf{x}(t) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

Classify the type and stability of the system's critical point at $(0, 0)$.

(b) Suppose one of the eigenvalues of the coefficient matrix $\mathbf{A}$ is $-4 + 5i$, which has a corresponding eigenvector $\begin{pmatrix} 1+i \\ -2i \end{pmatrix}$. Write down the system's real-valued general solution.

(c) Classify the type and stability of the critical point at $(0, 0)$ for the system described in (b).

(d) Suppose $\mathbf{A}$ has eigenvalues 2 and -3, classify the type and stability of the system's critical point at $(0, 0)$.