Name: ______________

Student Number: ______________

Instructor: ______________

Section: ______________

MATH 251

Midterm Exam II

July 20, 2006

This exam has 10 questions for a total of 100 points. There are **5** partial credit questions. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.**

**THE USE OF CALCULATORS IS NOT PERMITTED IN THIS EXAMINATION.** At the end of the examination, the booklet will be collected.

**Do not write in this box.**

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Total: ______________

1. (8 points) Consider a mass-spring system described by the differential equation

$$2u'' + \gamma u' + 8u = 2\sin(\omega t).$$

Answer the following questions.

(a) When $\gamma = 0$ what is the system's natural period, $T$?

(b) For what value(s) of $\gamma$ will the system be **critically damped**?

(c) If $\gamma = 4$, is the system underdamped or overdamped?

(d) (True of false) Resonance would occur if $\gamma = 0$ **and** $\omega = 2$.

2. (5 points) Suppose $y(t)$ is the solution of the second order linear initial value problem

$$y'' + 4y = t, \qquad y(0) = 1, \qquad y'(0) = 0.$$

What is the Laplace transform of $y(t)$?

(a) $Y(s) = \dfrac{1}{s^2(s^2+4)}$

(b) $Y(s) = \dfrac{1-s^2}{s^2(s^2+4)}$

(c) $Y(s) = \dfrac{s^2+1}{s^2(s^2+4)}$

(d) $Y(s) = \dfrac{s^3+1}{s^2(s^2+4)}$

3. (5 points) What is the inverse Laplace transform of $\dfrac{3s+4}{s^2+2s+5}$?

(a) $3e^t \cos 2t - \dfrac{7}{2}e^t \sin 2t$

(b) $3e^t \cos 2t + 2e^t \sin 2t$

(c) $3e^{-t} \cos 2t + \dfrac{1}{2}e^{-t} \sin 2t$

(d) $3e^{-t} \cos 2t + e^{-t} \sin 2t$

4. (5 points) Let $f(t) = 1 - u_1(t) + u_5(t)\,(t-3) + u_8(t)\,t^2$. Then $f(7) =$

(a) 2

(b) 4

(c) 68

(d) $f$ is undefined at $t = 7$.

5. (5 points) Which of the following systems of first order linear equations is equivalent to the second order linear equation

$$y'' - 2y' + 5y = 0?$$

(a) $\begin{cases} x_1' &= x_1 \\ x_2' &= 5x_1 - 2x_2 \end{cases}$

(b) $\begin{cases} x_1' &= x_1 \\ x_2' &= -5x_1 + 2x_2 \end{cases}$

(c) $\begin{cases} x_1' &= x_2 \\ x_2' &= 5x_1 - 2x_2 \end{cases}$

(d) $\begin{cases} x_1' &= x_2 \\ x_2' &= -5x_1 + 2x_2 \end{cases}$

6. (16 points)

   (a) (6 points) When solving the following nonhomogeneous equation using the method of undetermined coefficients, what is a suitable form of the prticular solution of $Y(t)$ to use? (**DO NOT ATTEMPT TO SOLVE FOR THE COEFFICIENTS!**)

$$y'' + 2y' - 3y = t^2e^t + e^{3t}\cos 2t - 3t\sin 3t.$$

   (b) (10 points) Find the general solution of the nonhomogeneous linear equation

$$y'' - 4y' + 3y = 3t^2 - 1.$$

7. (16 points) A mass of 2 $kg$ stretches a spring 2 $m$. The system has a damping constant of 4 $kg/s$. The mass is pulled 4 $m$ downward from its equilibrium position and released with zero initial velocity. You may take $g = 10\ m/s^2$ as the gravitational constant.

   (a) (12 points) Set up and solve an initial value problem to find the system's diplacement function $u(t)$.

   (b) (2 points) What is the quasi-frequency of the system?

   (c) (2 points) What is $\lim_{t\to\infty} u(t)$?

8. (12 points) Rewrite the following piecewise continuous function $f(t)$ in terms of the unit-step functions, and then find its Laplace transform.

$$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ 3t - 6, & 2 \le t < 4 \\ 4e^{-3t}, & 4 \le t \end{cases}.$$

9. (16 points) Use the Laplace transform to solve the initial value problem

$$y'' + 4y' + 8y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = -1.$$

**No credit will be given if the Laplace transform is not used to solve this problem.**

10. (12 points)

(a) (10 points) Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

(b) (2 points) What is $\lim_{t \to \infty} |\mathbf{x}(t)|$?