This exam has 9 questions for a total of 100 points. Show all your work! **In order to obtain full credit for partial credit problems, all work must be shown.** Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number.

**You may not use a calculator on this exam. Please turn off and put away your cell phone.**

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Do not write in this box.

| 1: ____ | (11) |
| 2: ____ | (11) |
| 3: ____ | (10) |
| 4: ____ | (10) |
| 5: ____ | (12) |
| 6: ____ | (11) |
| 7: ____ | (13) |
| 8: ____ | (14) |
| 9: ____ | (8) |

Total: ____
1. (11 points) Solve the following initial value problem. Give your answer in the explicit form.

\[ y' = \frac{3te^{-3t} - 2t}{4y}, \quad y(0) = -2. \]
2. (11 points) Consider the following equation
\[ t^2 y' - 2ty = 4t^3 \ln(t). \]

(a) (7 points) Find the general solution of the equation.

(b) (2 points) Find the solution satisfying the initial condition \( y(1) = -6 \).

(c) (2 points) What is the interval of validity for the solution found in (b)?
3. (10 points) Consider the differential equation

\[ 2x e^y + y^2 e^x + 6x + (x^2 e^y + 2ye^x - 6y^2)y' = 0. \]

(a) (3 points) Verify that it is an exact equation.

(b) (7 points) Find the solution of this equation satisfying \( y(3) = 0 \). You may leave your answer in an implicit form.
4. (10 points) Suppose a ball with mass 1 kg is thrown upward with initial speed of 10 m/sec from the roof of a building 40 meter high. Suppose it experiences air-resistance proportional to its speed, and the drag coefficient is \( k = 0.5 \). Assume \( g = 10 \, \text{m/sec}^2 \) and the positive direction is downward.

(a) (4 points) Set up an initial value problem that models the ball’s velocity while it is in the air.

(b) (4 points) Solve the initial value problem.

(c) (2 points) Suppose the building is next to a tall cliff such that the ball could free fall for a very long time. To what value does the ball’s velocity will approach, after a very long time, according to the model described in the problem?
5. (12 points) Consider the autonomous differential equation

\[ y' = y^2(y^2 + 2y - 3). \]

(a) (6 points) Find all of its equilibrium solutions. Classify the stability of each equilibrium solution. Justify your answer.

(b) (2 points) If \( y(0) = 1 \), then what is \( \lim_{t \to \infty} y(t) \)?

(c) (2 points) If \( y(1) = -1 \), what is \( \lim_{t \to \infty} y(t) \)?

(d) (2 points) If \( y(\pi) = \alpha \) and \( \lim_{t \to \infty} y(t) = 0 \), what are all the possible values of \( \alpha \)?
6. (11 points) Consider the second order linear equation
\[ y'' - y' - 6y = 0. \]

(a) (3 points) Find the general solution of the equation.

(b) (4 points) Find the solution satisfying the initial conditions \( y(1425) = 10, \ y'(1425) = 10. \)

(c) (4 points) Let \( y_1 \) be the solution satisfying initial conditions \( y(0) = 6, \ y'(0) = \beta. \) Suppose \( \lim_{t \to \infty} y_1(t) = 0. \) Find all the possible value(s) of \( \beta. \)
7. (13 points) Given that \( y_1(t) = t^2 + 1 \) and \( y_2(t) = t + 2 \) are both solutions of the second order homogeneous linear equation
\[
y'' + p(t)y' + q(t)y = 0.
\]
Answer each question below. State a brief reason that justifies each answer.

(a) (2 points) Find their Wronskian \( W(y_1, y_2)(t) \).

(b) (2 points) True or false: \( y_1 \) and \( y_2 \) form a set of fundamental solutions of this equation.

(c) (2 points) True or false: \( y_3(t) = t^2 + 2t + 5 \) is also a solution of the equation.

(d) (2 points) True or false: \( y_4(t) = 0 \) is also a solution of the equation.

(e) (2 points) True or false: \( y_5(t) = t + 1 \) is also a solution of the equation.

(f) (3 points) Consider now the following nonhomogeneous equation, whose corresponding homogeneous equation is given above:
\[
y'' + p(t)y' + q(t)y = g(t).
\]
Suppos \( Y = -3e^{-t} \) is a known solution of this nonhomogeneous equation, what is its general solution?
8. (14 points) Consider the second order nonhomogeneous linear equation

\[ y'' - 4y' = 3t - 6e^{-4t}. \]

(a) (3 points) Find \( y_c(t) \), the solution of its corresponding homogeneous equation.

(b) (7 points) Find its general solution by using the Method of Undetermined Coefficients.

(c) (4 points) What is the form of particular solution \( Y \) that you would use to solve the following equation using the Method of Undetermined Coefficients? **DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.**

\[ y'' - 4y' = 5t^2e^t \cos 2t - 2te^{-4t}. \]
9. (8 points) (a) Construct a second order homogeneous linear equation with constant coefficients, such that \( y_1 = \sqrt{2} e^{-\pi t} \) and \( y_2 = \sqrt{11} t e^{-\pi t} \) are two of its solutions.

(b) Suppose \( y_1(t) \) and \( y_2(t) \) are two solutions of a certain second order linear differential equation

\[
\sin(2t)y'' + \cos(2t)y' - t^2 y = 0, \quad 0 < t < \frac{\pi}{4}.
\]

According to the Abel’s Theorem, what is the general form of their Wronskian, \( W(y_1, y_2)(t) \)?